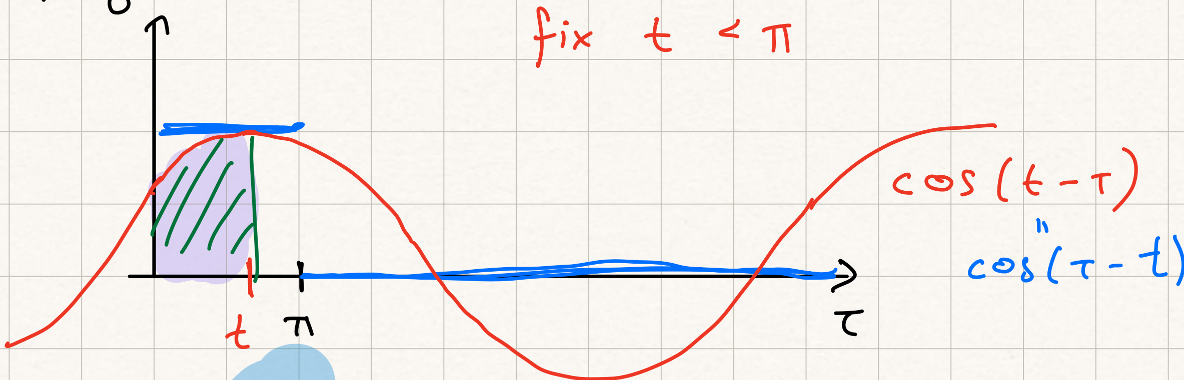
4, 5, 8, 9, 12

4. $f(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$ $\quad g(t) = \cos(t)$

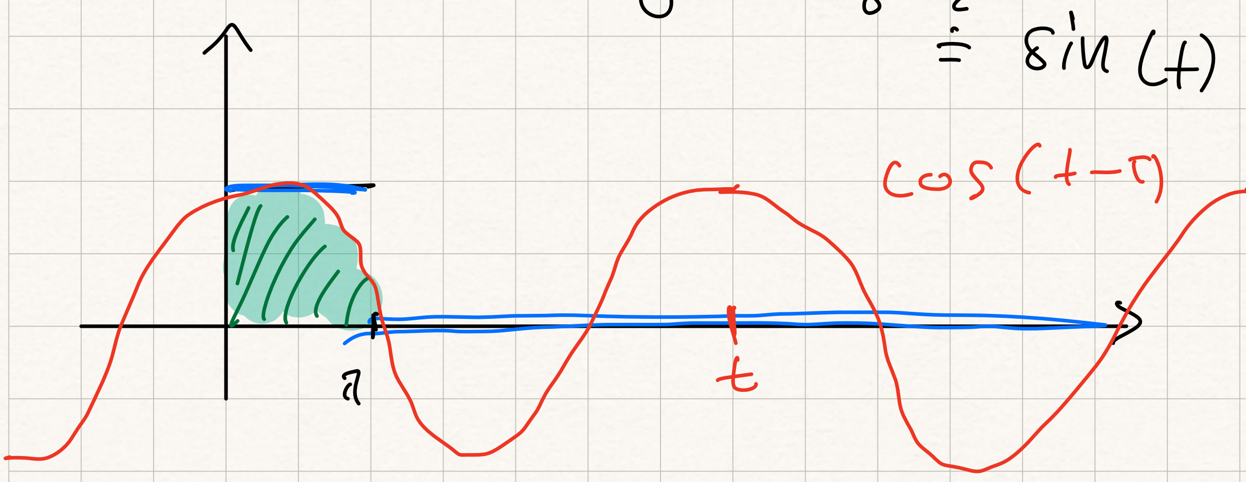
$f * g = ?$

fix $t < \pi$



$f * g = \int_0^t f(\tau)\cos(t-\tau)\,d\tau$

1) for $0 \le t < \pi$: $\quad f*g(t) = \int_0^t \cos(t-\tau)\,d\tau \stackrel{?}{=} \sin(t)$



$f * g = \int_0^t f(\tau)\cos(t-\tau)\,d\tau$

$$= \int_0^{\pi} \cos(t-\tau)\,d\tau \quad \text{if} \quad t > \pi$$

$$= 2\sin(t)$$

5.

$$f_\alpha(t) = \cos(\alpha t), \quad g(t) = \cos(t)$$

$$f_\alpha * g$$

$$f_\alpha * g = \mathcal{L}^{-1}\{\mathcal{L}\{f_\alpha * g\}\}$$

$$\overset{\text{conv. thm}}{=} \mathcal{L}^{-1}\{\mathcal{L}\{f_\alpha\}\mathcal{L}\{g\}\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\alpha^2}\,\frac{s}{s^2+1}\right\}$$

$s^2+\alpha^2 \to$ irreducible quadratic if $\alpha \neq 0$

Assume $\alpha \neq 0, 1$

Partial fractions:

$$\frac{s}{s^2+\alpha^2}\,\frac{s}{s^2+1} = \frac{A_1 s + B_1}{s^2+\alpha^2} + \frac{A_2 s + B_2}{s^2+1}$$

$$s^2 = (A_1 s + B_1)(s^2+1) + (A_2 s + B_2)(s^2+\alpha^2)$$

$s = \alpha i$

$$-\alpha^2 = (A_1 \alpha i + B_1)(1-\alpha^2)$$

Real pt: $\quad -\alpha^2 = B_1(1-\alpha^2) \Rightarrow \quad B_1 = \dfrac{\alpha^2}{\alpha^2-1}$

Im pt: $\quad A_1 = 0$

$s = i$

$$-1 = (A_2 i + B_2)(\alpha^2 - 1)$$

$$\Rightarrow \quad B_2 = \frac{-1}{\alpha^2-1}, \quad A_2 = 0$$

So:

$$\mathcal{L}^{-1} \frac{s^2}{(s^2+\alpha^2)(s^2+1)} = \mathcal{L}^{-1}\left\{\frac{\alpha^2}{\alpha^2-1}\,\frac{1}{s^2+\alpha^2} - \frac{1}{\alpha^2-1}\,\frac{1}{s^2+1}\right\}$$

$$= \frac{\alpha}{\alpha^2-1}\sin(\alpha t) - \frac{1}{\alpha^2-1}\sin(t)$$

$$= f_\alpha * g\,(t) \quad \text{if } \alpha \neq 1, 0.$$

If $\alpha = 0$ $\quad f_\alpha * g = 1 * \cos(t) = \ldots = \sin(t)$

If $\alpha = 1$

$$f_1 * g = \mathcal{L}^{-1}\left\{ \mathcal{L}\{\cos(t)\}^2 \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s^2}{(s^2+1)^2} \right\}$$

$$\overset{\text{table}}{=} \frac{1}{2}(\sin t + t\cos t)$$

8. $$\mathcal{L}^{-1}\left\{ \frac{1}{(s^2+k^2)^2} \right\} = \mathcal{L}^{-1}\left\{ -\frac{1}{2s} \underbrace{\frac{d}{ds}\frac{1}{(s^2+k^2)}}_{-\frac{1}{(s^2+k^2)^2}2s} \right\}$$

$$\overset{\text{Rule}}{=} -\frac{1}{2}\int_0^t \mathcal{L}^{-1}\left\{ \frac{d}{ds}\frac{1}{s^2+k^2} \right\} d\tau$$

$$\overset{\text{Rule}}{=} -\frac{1}{2}\int_0^t (-\tau)\, \mathcal{L}^{-1}\left\{ \frac{1}{s^2+k^2} \right\} d\tau$$

$$= \frac{1}{2k}\int_0^t \tau \sin(k\tau)\, d\tau \overset{\text{IBP}}{=} \ldots$$

9. $$\frac{di}{dt} + 150\,i + \frac{1}{2}10^4 \int_0^t i(\tau)\, d\tau = e(t)$$

$i(0) = 0$

Take $\mathcal{L}$

$$sI(s) - \cancel{i(0)}^{0} + 150\, I(s) + \frac{1}{2}10^4 \frac{I(s)}{s}$$

$$= \mathcal{L}\{e(t)\}$$

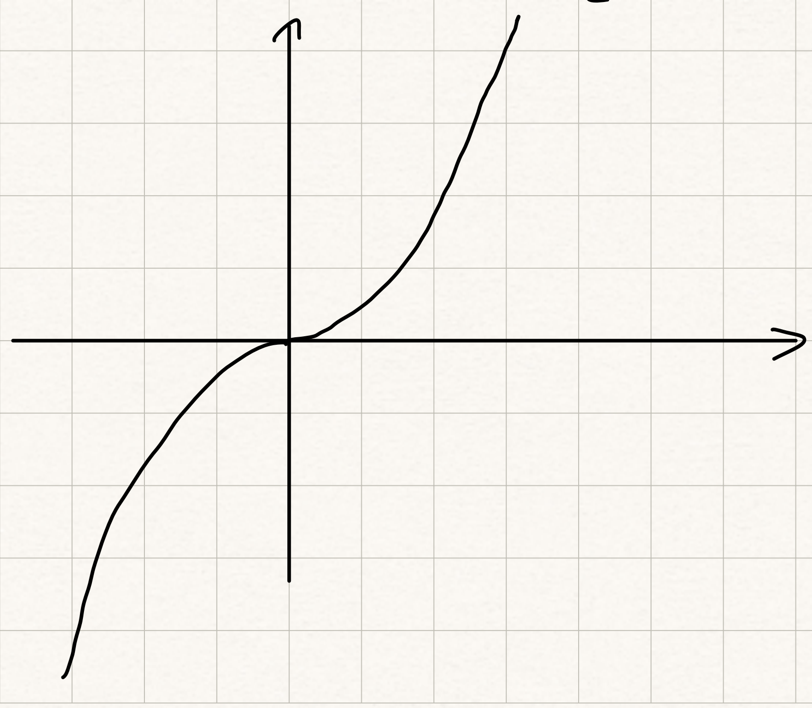
$$\Rightarrow I(s) = \frac{1}{s + 150 + \frac{1}{2}10^4 \frac{1}{s}} \mathcal{L}\{e(t)\} \quad \ldots$$

compute $\mathcal{L}\{e(t)\}$, partial fractions.

12. $f_1(t) = \tan(t) = \dfrac{\sin(t)}{\cos(t)}$

$$f_1(t + 2\pi) = \frac{\sin(t+2\pi)}{\cos(t+2\pi)} = \frac{\sin(t)}{\cos(t)} = f_1(t)$$

b) $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$



$f_4 = \arctan(t+1)$