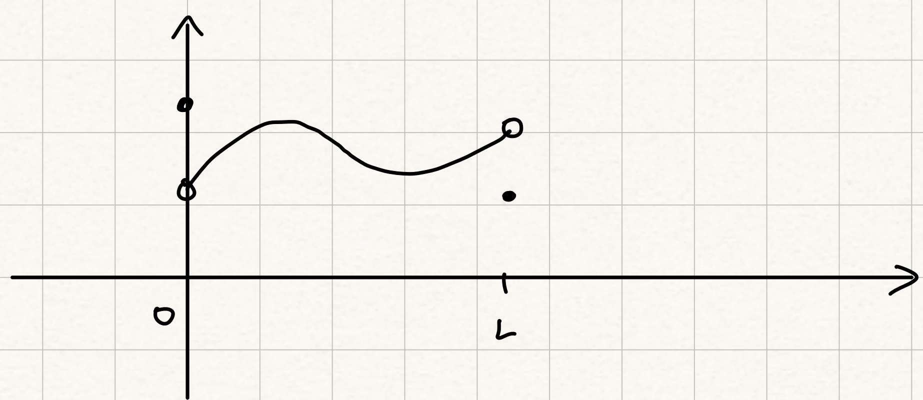
Lesson 31 04/04/2022

Fourier sine & cosine series

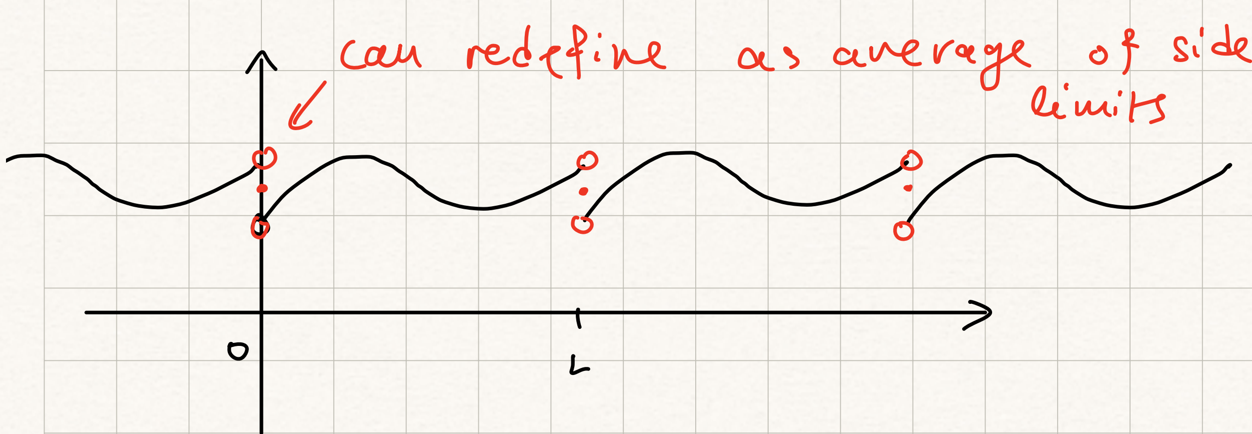
Given: piecewise smooth function $f(t)$ defined on an interval $[0,L]$

Want: Use Fourier series to analyze.

Issue: F.S. needs a periodic function and we do not have one.



One way to do it:



Can take F.S. of this, it will converge

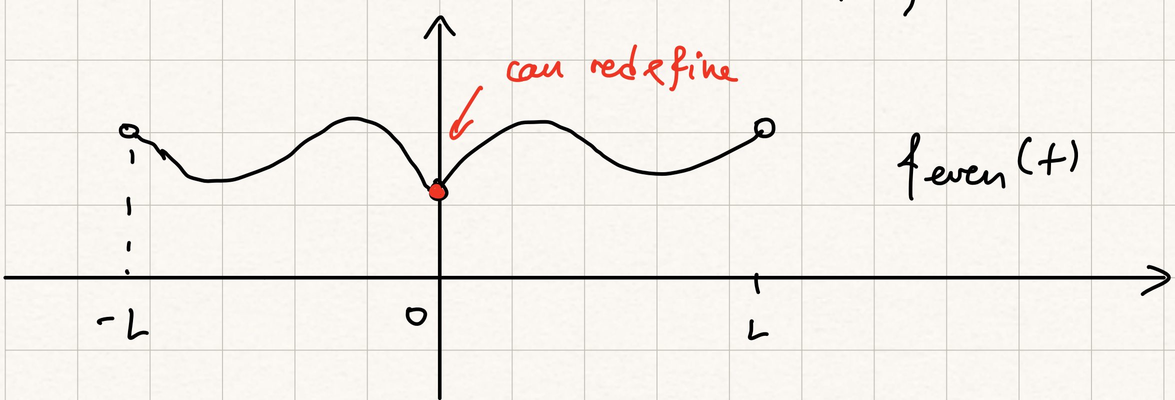
to f on $[0,L]$ (or average of its side limits)

Not optimal way. Instead:

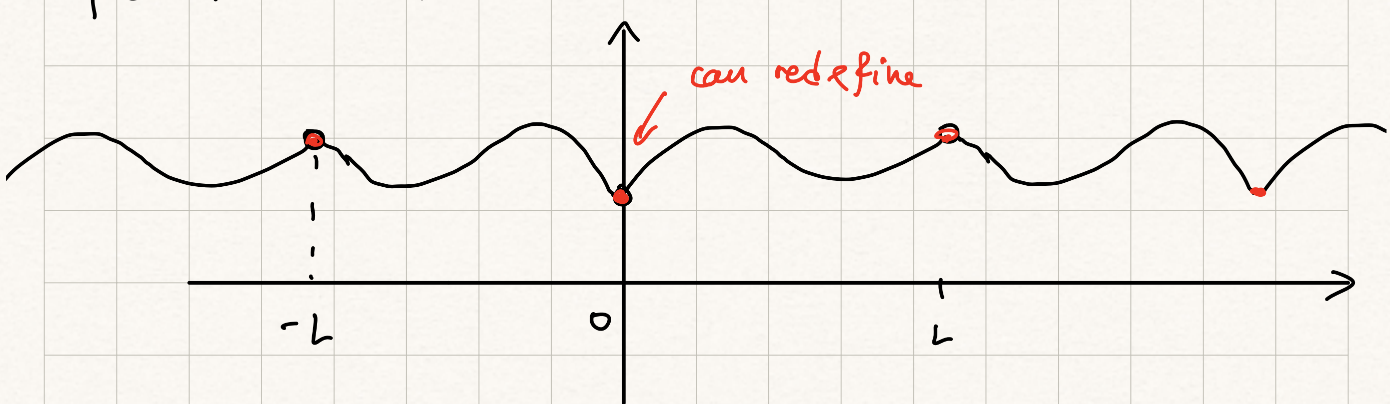
Step 1: Extend f to interval $[-L, L]$.

2 natural ways: $\begin{cases} \text{Case I: as an even function} \\ \text{Case II: as an odd function.} \end{cases}$

Case I: Even extension of f on $[-L, L]$:



Step 2: Extend f_{even} as a $2L$-periodic function on $\mathbb{R}$.



A $2L$-periodic function: can write F.S.

$$f_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

Even fct $\Rightarrow$ no sine terms!

$$a_0 = \frac{2}{L}\int_0^L \underbrace{f_{even}(t)}_{= f(t)}\,dt, \quad a_n = \frac{2}{L}\int_0^L \underbrace{f_{even}(t)}_{= f(t)}\cos\left(\frac{n\pi}{L}t\right)dt$$

In $[0,L]$: $f_{even}(t) = f(t)$

The Fourier Cosine series of a piecewise smooth function $f(t)$ defined on $[0,L]$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt, \quad a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)dt.$$

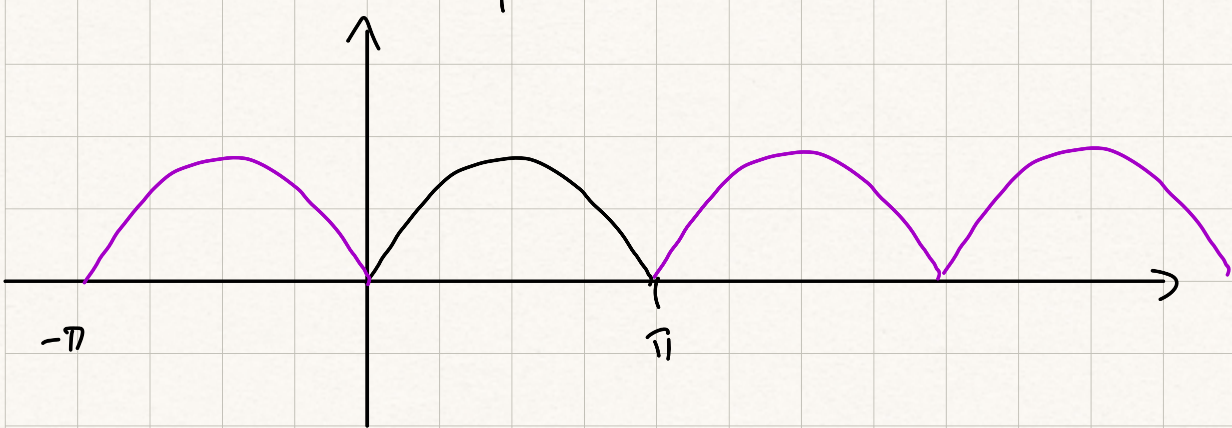
So: F. Cosine series is the usual F.S. of the even extension of f.

Note: F.C.S. converges to f (or

average of side limits at discont.)
on interval $[0,L]$, to f_{even} on $\mathbb{R}$.

Ex: $f_1(t) = \sin(t)$, $t \in [0,\pi]$

1: Draw even periodic extension



Fourier cosine series:

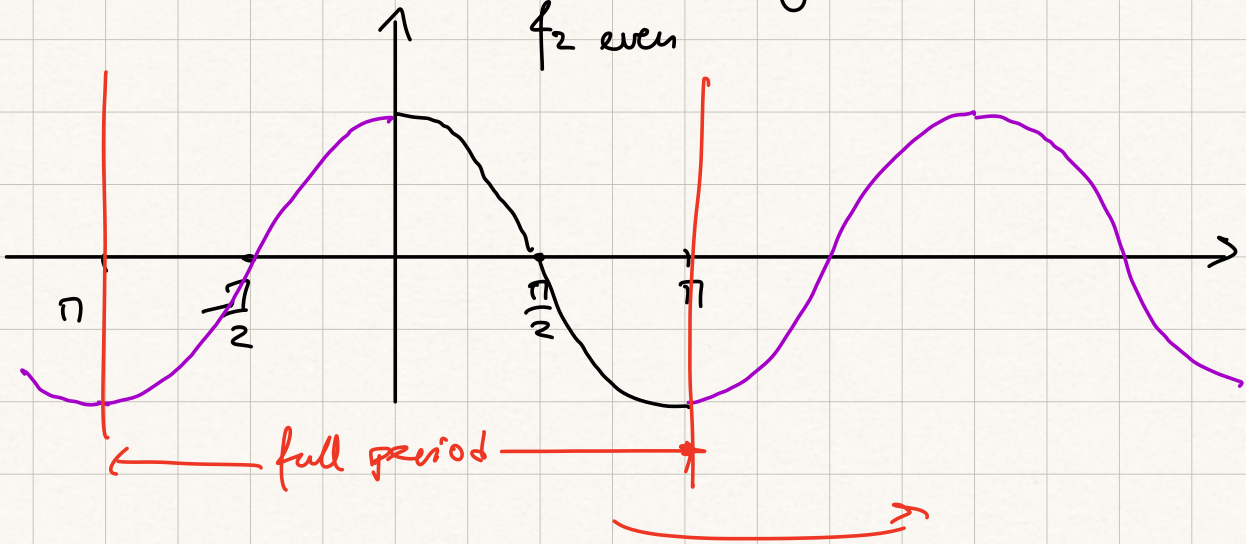
$$f_1 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{2}{\pi}\left(-\cos(t)\right)\Big|_0^{\pi} = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt = \_\_\_ \quad \text{HW}$$

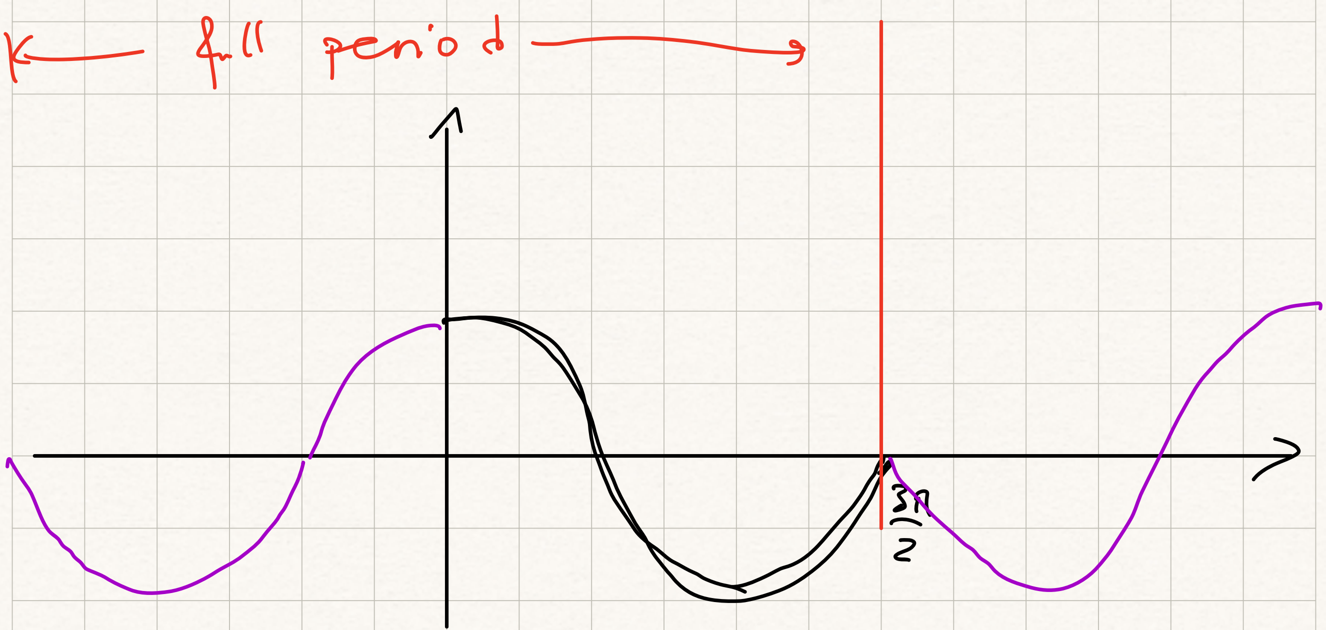
Ex: $f_2(t) = \cos(t)$ on $[0,\pi]$
$f_3(t) = \cos(t)$ on $[0, \frac{3\pi}{2}]$

Draw even extensions, extended to $\mathbb{R}$.
(Different wording: what does F. cosine series of f_2, f_3 converge to?)



Notice: $f_{2\,\text{even}} = \cos(t)$ on $\mathbb{R}$

So cosine series of $f_2(t)$ has only one term: $f_2(t) = \cos(t)$



f_3 is not the usual $\cos(t)$ on $\mathbb{R}$, F. cosine series has ∞ many terms.

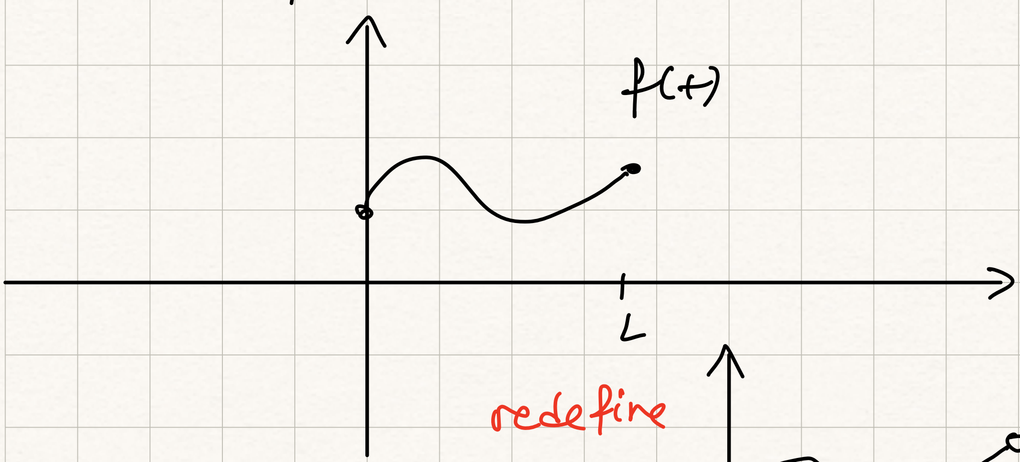
$$f_3(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n2}{3}t\right)$$

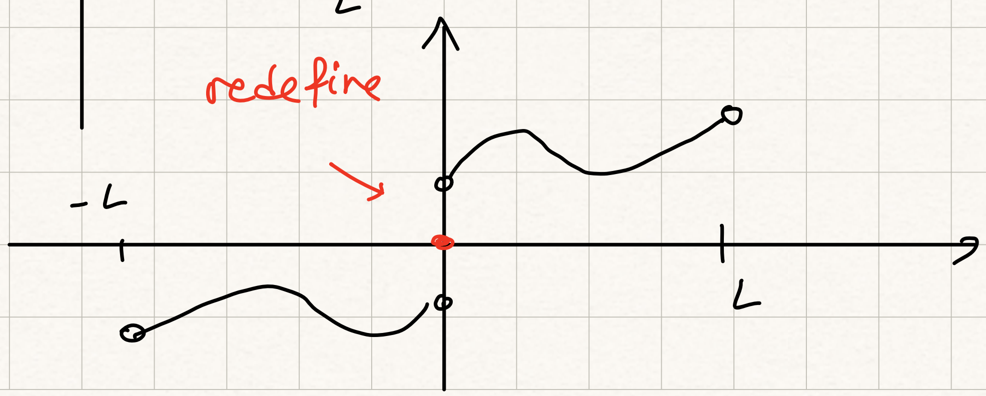
$$L = \frac{3\pi}{2}$$

$$a_n = \frac{4}{3\pi} \int_0^{\frac{3\pi}{2}} \cos(t) \cos\left(\frac{n \cdot 2}{3} t\right) dt$$

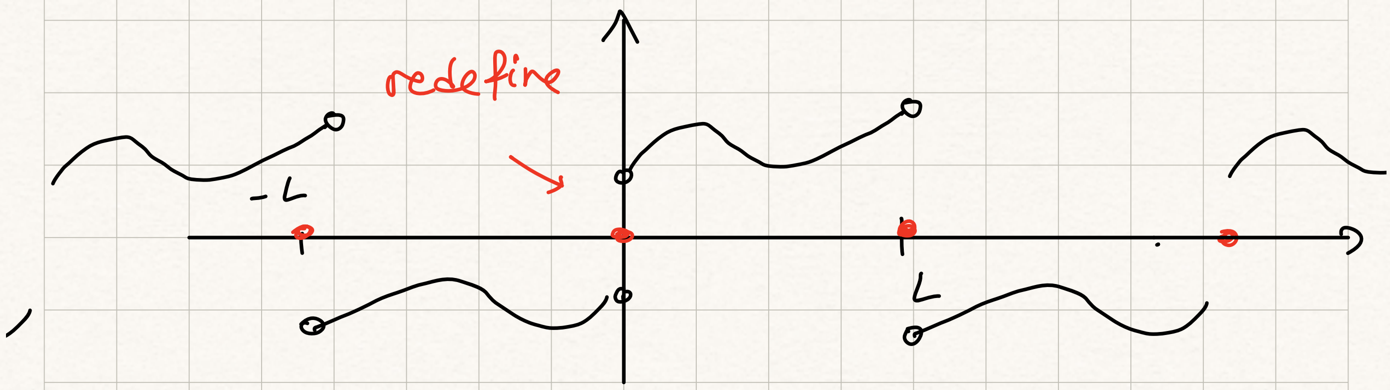
$$= \ldots$$

Saw: how to use an even extension to produce a F. series. Can do same w/ odd extension.





Extend periodically:



f_{odd} extended on $\mathbb{R}$.

Fourier series of f_{odd} is by def'n the Fourier sine series of f:

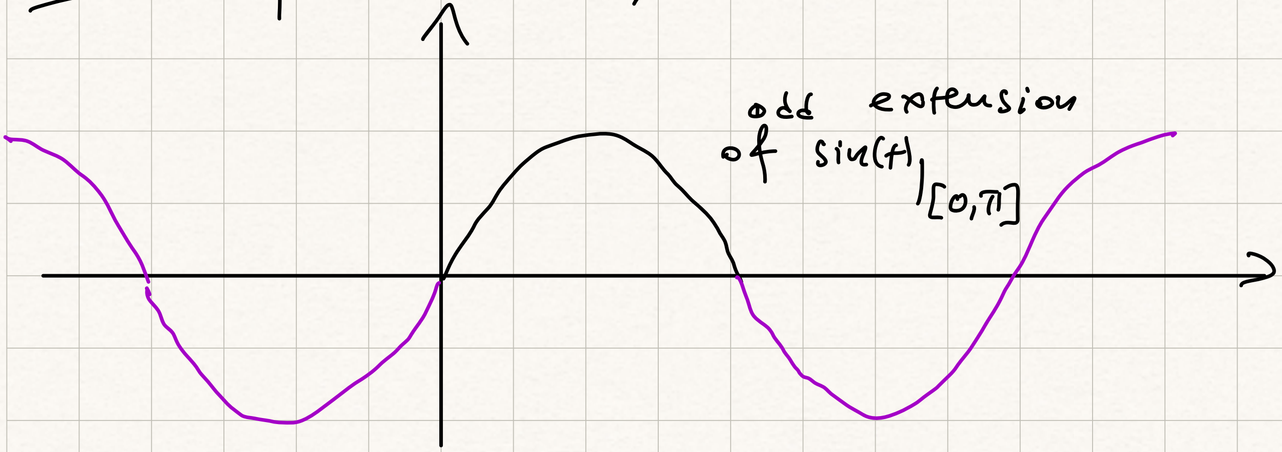
$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right)$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

It converges to f on $[0,L]$, to f_{odd} on $\mathbb{R}$.

Note: F. sine & cosine series converge to different functions on $\mathbb{R}$ but to the same function on $[0,L]$.

Ex: $f(t) = \sin(t)$, $t \in [0, \pi]$



This is the usual $\sin(t)$ on $\mathbb{R}$. Fourier sine series has only one term, $\sin(t)$.