# Lesson 32

04/06/22

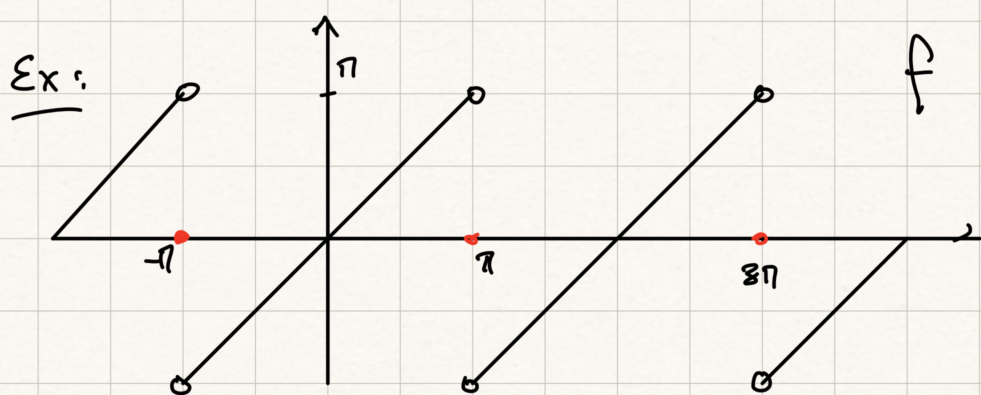
## 9.3

### Endpoint problems w/ Fourier series

**Hope:** differentiate F.S. term by term

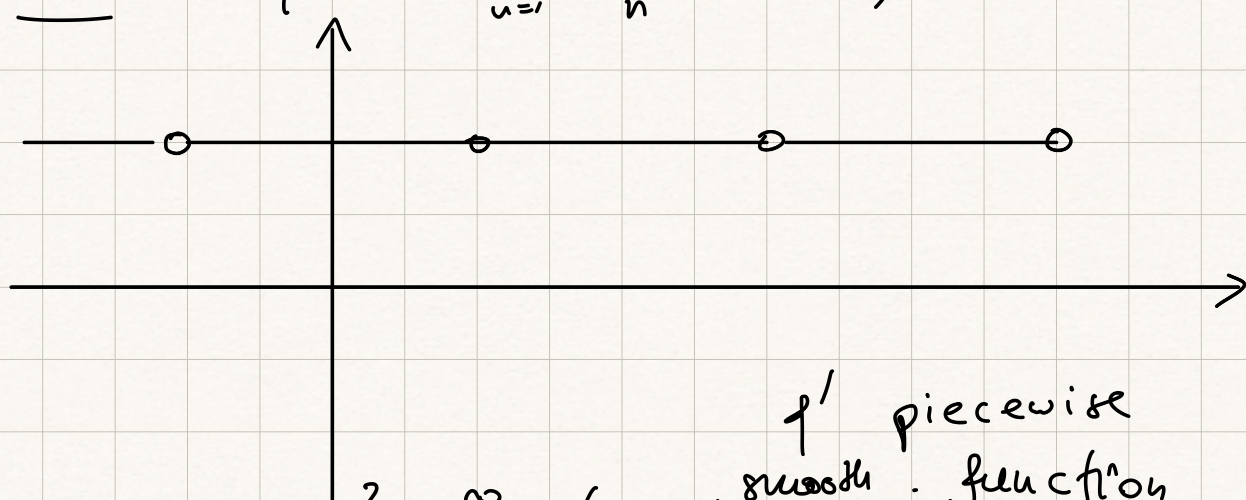
**Issue:** Does not always work.

**Ex:**




$f$ piecewise smooth

**Seen:** $f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nt)$



$f'$ piecewise smooth function

**Hope:** $f' \stackrel{?}{=} \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n} \sin(nt) \right)'$

$$\left(\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\sin(nt)\right)\right)' = \sum_{n=1}^{\infty} 2(-1)^{n+1}\cos(nt)$$

At $t=0$

$$\sum_{n=1}^{\infty} 2(-1)^{n+1}$$

and this doesn't converge!

<u>Thm:</u>
- If $f$ cont. for all $t$
- $2L$ periodic
- $f'$ piecewise smooth ($f'$, $f''$ piecewise cont.)

<u>then</u> F.S. of $f$ can be differentiated term by term:

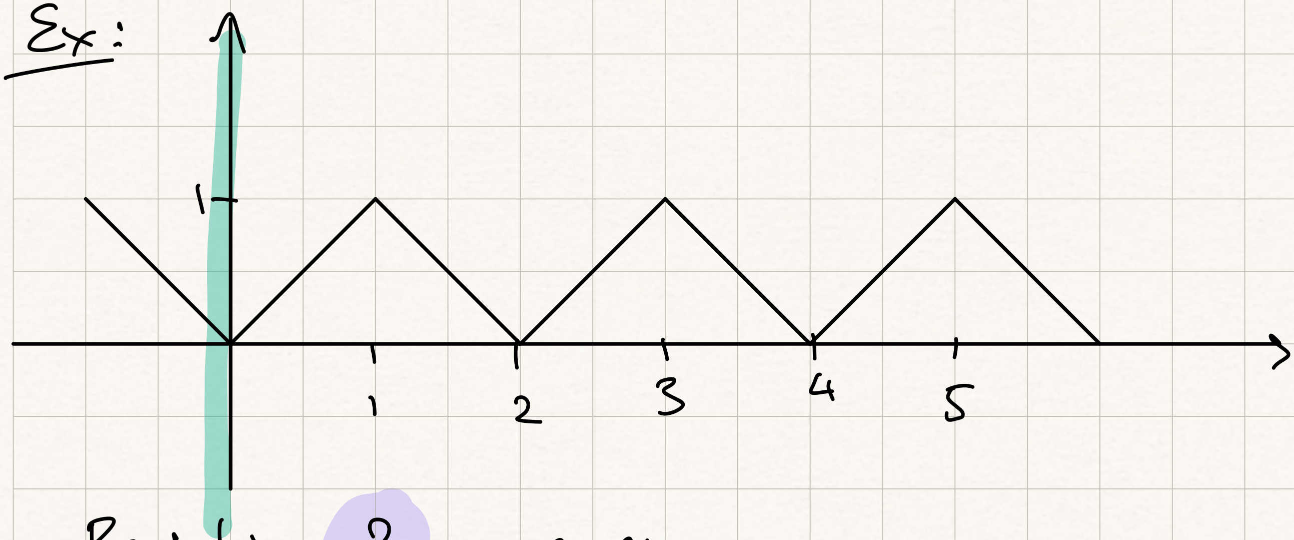
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}t\right) + b_n\sin\left(\frac{n\pi}{L}t\right)\right)$$

then

$$f'(t) = \sum_{n=1}^{\infty} a_n\left(-\frac{n\pi}{L}\sin\left(\frac{n\pi}{L}t\right)\right) + b_n\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}t\right)$$

at pts where $f'$ is cont, otherwise series conv. to average of side limits.

Ex:




Period: 2, even.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{1} t\right)$$

$$a_0 = \frac{2}{1}\int_0^1 t\,dt = 2\,\frac{t^2}{2}\Big|_0^1 = 1$$

$$a_n = 2\int_0^1 t\cos(n\pi t)\,dt \qquad dv =$$

$$= \frac{2}{n\pi}\int_0^1 \overset{u}{t}\,\big(\overbrace{\sin(n\pi t)}^{v}\big)'\,dt$$

$$= \underbrace{\frac{2}{n\pi}\, t\sin(n\pi t)\Big|_0^1}_{0} - \frac{2}{n\pi}\int_0^1 \sin(n\pi t)\,dt$$

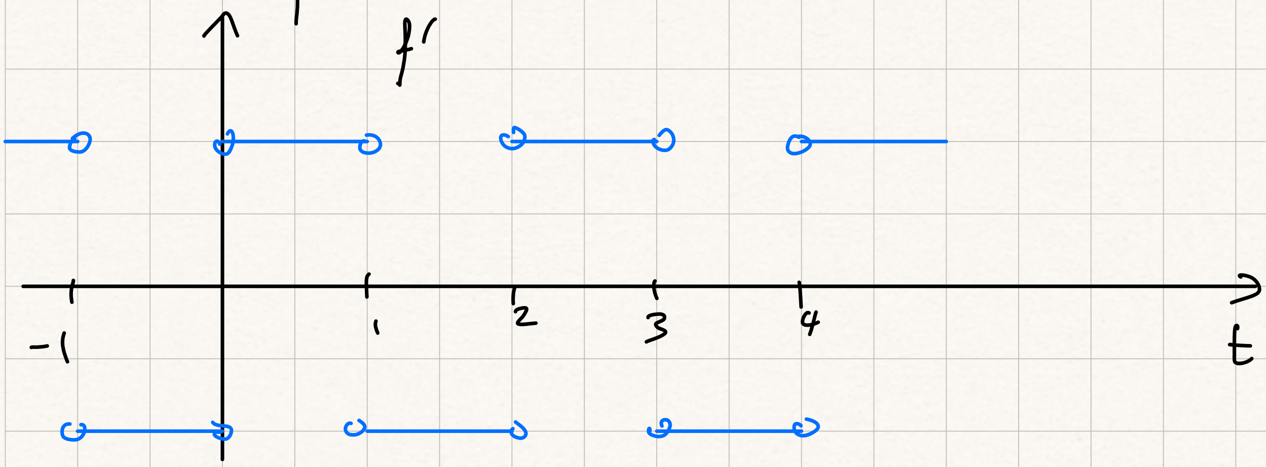
$$= +\frac{2}{(n\pi)^2}\cos(n\pi t)\Big|_0^1$$

$$= \frac{2}{(n\pi)^2}\left(\cos(n\pi) - 1\right)$$

So:

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left(\cos(n\pi)-1\right)\cos(n\pi t). \quad (*)$$

Note: $f$ cont.




$f'$ is odd. $f'$ piecewise smooth.
The theorem applies. Check that it is the case: take F. S. of $f'$:

$$f' = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

$$b_n = 2\int_0^1 1 \sin(n\pi t)\,dt = -\frac{2}{n\pi}\cos(n\pi t)\Big|_0^1$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi)-1\right)$$

So:

$$f' = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\left(\cos(n\pi)-1\right)\right)\sin(n\pi t) \quad (**)$$

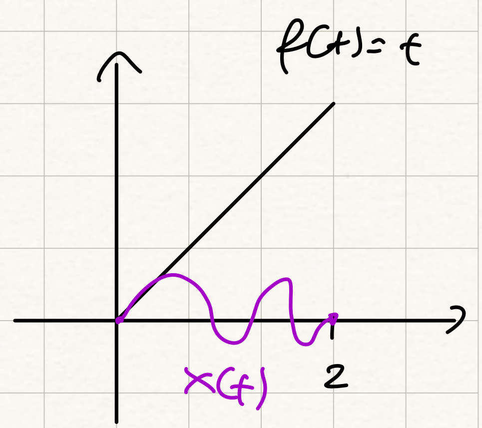
The terms of $(**)$ are derivatives of terms of $(*)$

Solve endpt problems using F.S.

$$\text{I}: \begin{cases} x'' + cx = f(t) & \text{on } (0,L) \\ x(0) = x(L) = 0 \end{cases}$$

$$\text{II}: \begin{cases} x'' + cx = f(t) & \text{on } (0,L) \\ x'(0) = x'(L) = 0 \end{cases}$$

Hope to find: $x(t)$ in the form of a series expansion.




Ex: $\begin{cases} x'' + 2x = t & \text{on } (0,2) \\ x(0) = x(2) = 0 \quad (*) \end{cases}$

Step 1:

Extend $f(t) = t$ to be periodic: even or odd?

bec. of $(*)$, take odd extension.

not obvious yet.

Compute F.S. of odd ext, i.e. F. sine series of $f$.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}t\right) \qquad b_n = \frac{2}{2}\int_0^2 t\sin\left(\frac{n\pi}{2}t\right)dt$$

Exercise: $b_n = \dfrac{2}{\pi n}\left(1-\cos(\pi n)\right)$

<u>Step 2</u>-. Assume that $x(t)$ has a F. series expansion w/ period $2L = 4$

$$x(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}\left(\tilde{a}_n \cos\left(\frac{n\pi}{2}t\right) + \tilde{b}_n \sin\left(\frac{n\pi}{2}t\right)\right)$$

($\sim$ is to distinguish from coef. of $f(t)$)

Assume $x$ is nice enough that we can take 2 $t$-derivatives term by term.

<u>Step 3</u>: Compute $x''$ and plug into $x'' + 2x = t$.

$$x''(t) = \sum_{n=1}^{\infty}\left(\tilde{a}_n\left(-\left(\frac{n\pi}{2}\right)^2\right)\cos\left(\frac{n\pi}{2}t\right) + \tilde{b}_n\left(-\left(\frac{n\pi}{2}\right)^2\right)\sin\left(\frac{n\pi}{2}t\right)\right)$$

$$x'' + 2x = t$$

$$\Rightarrow \overbrace{\sum_{n=1}^{\infty}\left(\tilde{a}_n\left(-\left(\frac{n\pi}{2}\right)^2\right)\cos\left(\frac{n\pi}{2}t\right) + \tilde{b}\left(-\left(\frac{n\pi}{2}\right)^2\right)\sin\left(\frac{n\pi}{2}t\right)\right)}^{x''}$$

$$+ 2\frac{\tilde{a}_0}{2} + 2\sum_{n=1}^{\infty}\left(\tilde{a}_n \cos\left(\frac{n\pi}{2}t\right) + \tilde{b}_n \sin\left(\frac{n\pi}{2}t\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1-\cos(n\pi)\right)\sin\left(\frac{n\pi}{2}t\right)$$

Match coef. of $\cos\left(\frac{n\pi}{2}t\right)$, $\sin\left(\frac{n\pi}{2}t\right)$ & constant.

$\cos\left(\frac{n\pi}{2}t\right)$: $\quad \left(-\left(\frac{n\pi}{2}\right)^2 \tilde{a}_n + 2\tilde{a}_n\right) = 0$

$$\Rightarrow \underbrace{\left(2 - \left(\frac{n\pi}{2}\right)^2\right)}_{\neq 0} \tilde{a}_n = 0$$

$$\rightarrow \tilde{a}_n = 0$$

const: $\quad \tilde{a}_0 = 0$

$\sin\left(\frac{n\pi}{2}t\right)$ $\quad -\left(\frac{n\pi}{2}\right)^2 \tilde{b}_n + 2\tilde{b}_n = \frac{2}{\pi n}\left(1-\cos(n\pi)\right)$

$$\Rightarrow \tilde{b}_n = \frac{1}{2-\left(\frac{\pi n}{2}\right)^2}\,\frac{2}{\pi n}\left(1-\cos(n\pi)\right)$$

So: $\quad x(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi}{2}t\right).$ (*)

If $x'' + cx = f$ and $f$ has only sine terms, $x$ has only sine terms (*)

Note: $x(0) = 0 = x(2)$

So $*$ is a formal solution for our problem!