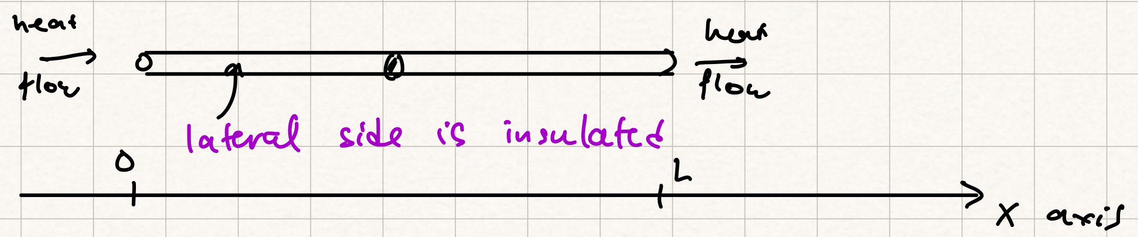
Lesson 34 04/11/22

9.5 Heat Conduction

Given: thin rod



Want to model: temperature of particles in the rod as time evolves.

Assume: temperature u is constant in each cross-section of rod

$$u = u(x, t)$$

position on x axis

time

Differential Eq'n:

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

Heat eq'n in 1 space dim'n

partial derivatives : partial diff'l eq'n (PDE)

$k > 0$

$k = \dfrac{K}{c\delta}$

K: thermal conductivity

c: specific heat (amount of heat needed to raise temp. of

1 gram by 1 degree C

δ: density of rod.

By analogy w/ ODEs where were given conditions such as $x(0)=0=x'(0)$ or $x(0)=x(L)=0$, we will need additional conditions to pinpoint a solution.

In our case: initial condition will be a function of the space variable.

$$u(x,0) = f(x)$$

temperature at different points of the rod at time $t=0$.

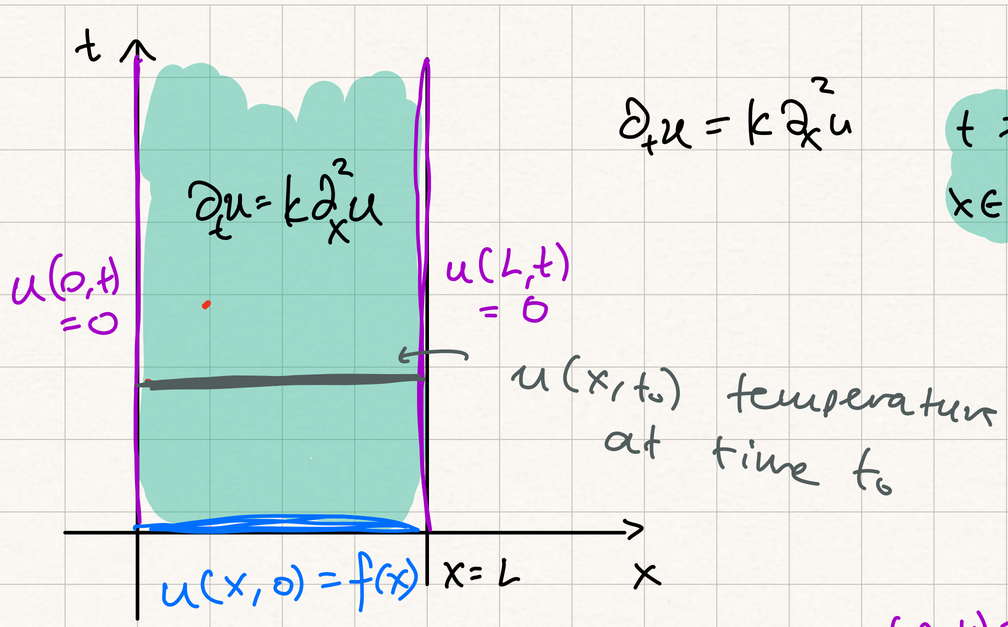
Endpoint conditions: ex. fixed temperature at endpts

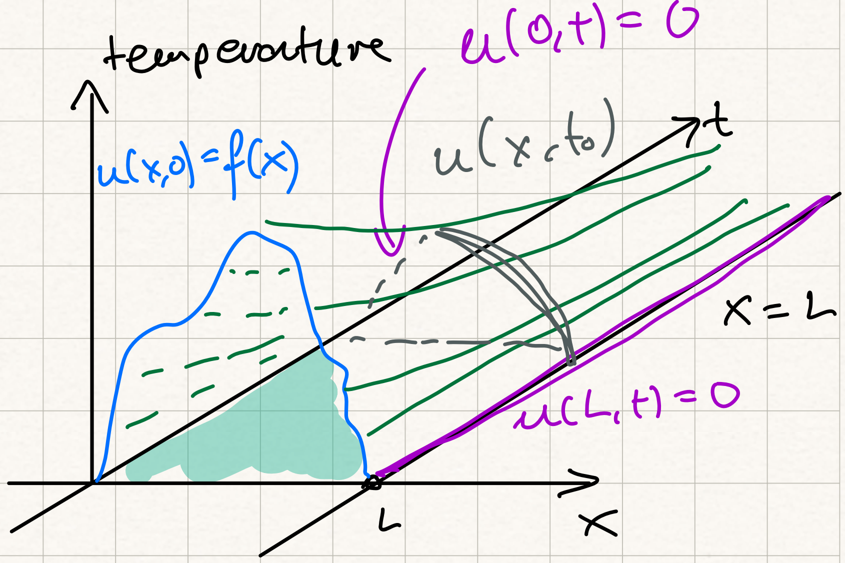
$$u(0,t) = u(L,t) = 0 \quad \text{for} \quad t > 0$$

$$\partial_t u = k\partial_x^2 u$$

$t > 0$

$x \in (0,L)$





In summary:

$$\begin{cases} u_t = k u_{xx} & x \in (0,L),\ t > 0 \quad \text{①} \\ \underline{u(0,t) = u(L,t) = 0} & t > 0 \quad \text{②} \\ \underline{u(x,0) = f(x)} & x \in (0,L) \quad \text{③} \end{cases}$$

$f(x)$: known, k known, u unknown

② Homogeneous condition (means it's 0)

③ Non-homogeneous condition (means it's not 0)

Observations: ① is linear u_1, u_2 sol's then $c_1 u_1 + c_2 u_2$ solves ①

$$\begin{aligned} \partial_t(c_1 u_1 + c_2 u_2) &= c_1 \partial_t u_1 + c_2 \partial_t u_2 \\ &= c_1 \, k \partial_x^2 u_1 + c_2 \, k \partial_x^2 u_2 \\ &= k\left(\partial_x^2 (c_1 u_1 + c_2 u_2)\right) \end{aligned}$$

c_1, c_2 const.

② linear: u_1, u_2 satisfy ②, $c_1 u_1 + c_2 u_2$ also does.

③ not linear:

$$u_1(x,0) = f(x)$$
$$u_2(x,0) = f(x)$$
$$(u_1 + u_2)(x,0) = 2f(x)$$

Strategy: find building blocks $u_1, u_2, u_3 ----$ (inf many) satisfying ① & ②

$$\begin{cases} \partial_t u_j = k \partial_x^2 u_j & j = 1, 2, -- \\ u_j(0,t) = u_j(L,t) = 0, & j = 1, 2, -- \end{cases}$$

Write infinite sum:

$$u(x,t) = \underbrace{\sum_{n=1}^{\infty} c_j u_j(x,t)}_{\text{should satisfy ①, ②}} \qquad c_j \text{ TBD}$$

Arrange c_j:

$$u(x,0) = \sum_{n=1}^{\infty} c_j u(x,0) = f(x).$$

If f piecewise smooth, sum converges to a sol'n, sol'n unique.

Preparation:

Method of characteristics

ODE: $x'' + ax' + bx = 0$, a, b const.

Char. eqn: $r^2 + ar + b = 0$, solve, $r = r_1, r_2$

→ If r_1, r_2 real, distinct

$$Ae^{r_1 t} + Be^{r_2 t}$$

→ If $r_1 = r_2$ $\quad Ae^{r_1 t} + Bte^{r_1 t}$

→ If $r_{1,2} = s \pm i\omega$ $\quad Ae^{st}\cos(\omega t) + Be^{st}\sin(\omega t)$

Exercise: $L > 0$

$$\textcircled{*} \begin{cases} x'' + \lambda x = 0 \\ x(0) = x(L) = 0 \end{cases} \qquad \lambda \text{ const.}$$

3 cases:

I. $\lambda = 0$

II. $\lambda = -\alpha^2$, $\alpha > 0$

III. $\lambda = \alpha^2$, $\alpha > 0$

Are there any sols of $\circledast$ not identically 0?

I: $\lambda = 0$ $\quad x'' = 0$

$$\Rightarrow x = A + Bt$$

$$x(0) = 0 \Rightarrow A = 0$$

$$x(L) = 0 \Rightarrow BL = 0 \Rightarrow B = 0$$

bec. $L > 0$.

II: $\lambda = -\alpha^2$

$$x'' - \alpha^2 x = 0 \qquad r^2 - \alpha^2 = 0$$

$$r = \pm\alpha$$

$$x(t) = Ae^{\alpha t} + Be^{-\alpha t}$$

$$x(0) = 0 \Rightarrow A + B = 0$$

$$x(L) = 0 \Rightarrow Ae^{\alpha L} - Ae^{-\alpha L} = 0$$

$$A\underbrace{e^{\alpha L}}_{} = A\underbrace{e^{-\alpha L}}_{}$$

$$\rightarrow A = 0$$

$$B = 0.$$