Lesson 35

04/13/22

Exercise: $\begin{cases} x'' + \lambda x = 0 \\ x(0) = x(L) = 0 \end{cases}$ (*)

$\lambda = 0$, $\lambda = -\alpha^2$, $\lambda = \alpha^2$

Looked for non-trivial sol's of (*)

Saw: $\lambda = 0$, $\lambda = -\alpha^2$, none

$\lambda = \alpha^2$, $\alpha > 0$

Find gen. sol'n, plug in endpt conditions

$x(t) = A\cos(\alpha t) + B\sin(\alpha t)$

$x(0) = 0 \Rightarrow A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$

$x(L) = 0 \Rightarrow B\sin(\alpha L) = 0$

true for $B \neq 0$ provided

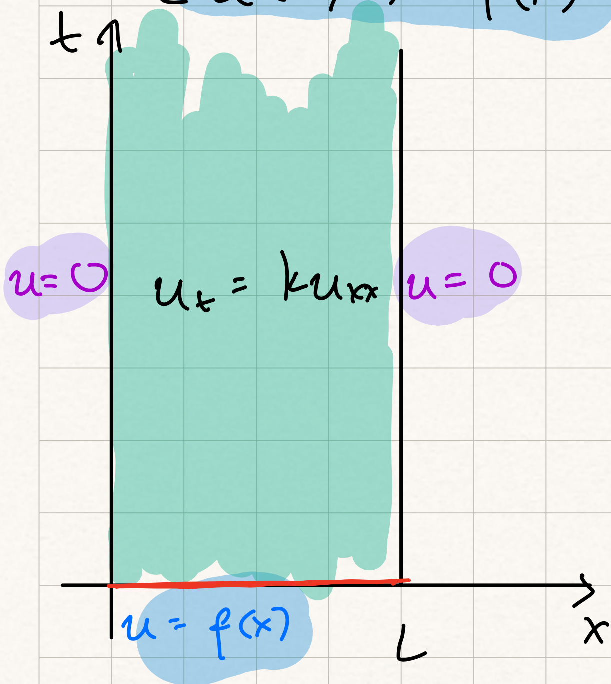
$\alpha L = n\pi \Rightarrow \alpha = \frac{n\pi}{L}$

$n = 1, 2, \ldots$

$$\boxed{\lambda = \left(\frac{n\pi}{L}\right)^2}.$$

Why we are doing this:

$$\begin{cases} u_t = k u_{xx} & x \in (0,L),\ t>0 & (1) \\ u(0,t) = u(L,t) = 0 & t>0 & (2) \\ u(x,0) = f(x) & x \in (0,L) & (3) \end{cases}$$



L, k, $f(x)$ given

Looking for u

Said: $$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

u_n: building blocks satisfying (1), (2)
At the end: arrange c_n so that (3) is satisfied.

Separation of Variables

Educated Guess: building blocks have special form:

$$u_n(x,t) = X_n(x)\,T_n(t) \qquad (*)$$

(1) $$\partial_t u_n = k\,\partial_x^2 u_n$$

$$X_n(x)\,T_n'(t) = k\,X_n''(x)\,T_n(t)$$

Separate => Variables

$$\underbrace{\frac{X_n''(x)}{X_n(x)}}_{\text{only depends on } x} = \underbrace{\frac{T_n'(t)}{k\,T_n(t)}}_{\text{only depends on } t}$$

→ Both are constant!

So: $$\frac{X_n''(x)}{X_n(x)} = -\lambda_n \qquad \text{(a constant)}$$

$$\frac{T_n'(t)}{k\,T_n(t)} = -\lambda_n \qquad \text{(same constant)}$$

$$\left.\begin{cases} X_n''(x) + \lambda_n X_n(x) = 0 \\ T_n'(t) = -\lambda_n k\,T_n(t) \end{cases}\right\} \lambda_n \text{ TBD}$$

(*) + (1) helped us turn the PDE into two ODEs.

$\to$ $u_n(x,t)$ should also satisfy ② :

$$u_n(0,t) = u_n(L,t) = 0 \quad \text{for} \quad t>0$$

$$\overset{*}{\Rightarrow} \quad X_n(0)\,T_n(t) = X_n(L)\,T_n(t) = 0 \qquad t>0$$

$$\overset{T_n \neq 0}{\Rightarrow} \qquad X_n(0) = X_n(L) = 0$$

L known, λ_n TBD

$$\begin{cases} X_n'' + \lambda_n X_n = 0 \\ X_n(0) = X_n(L) = 0 \end{cases}$$

$\Rightarrow$ X_n non-trivial exactly when

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$

So: $\boxed{X_n(x) = B_n \sin\left(\frac{n\pi}{L} x\right), \quad n = 1, 2, \ldots}$

(by exercise in the beginning)

(We can assume $B_n = 1$)

To find T_n:

$$T_n' = -k\underbrace{\left(\frac{n\pi}{L}\right)^2}_{=\lambda_n} T_n \Rightarrow \boxed{T_n = C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

(can assume $c_n = 1$)

So: $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$

Recall:

3 $\quad u(x,0) = f(x)$

or $\quad \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$

F. Sine series for f:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$\Rightarrow$ take $c_n = b_n =$

In summary: sol'n 1 - 3

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}}$$

Notice: green term $\to 0$ as $t \to \infty$

Ex: $\begin{cases} \partial_t u = \partial_x^2 u & 0 < x < 5 \\ u(0,t) = u(5,t) = 0 \\ u(x,0) = 25 \end{cases}$

Check: F. sine series coef. $f(x) = 25$ on $[0,5]$

$$b_n = 50 \frac{(-1)^n - 1}{\pi n}$$

$$u(x,t) = \sum_{n=1}^{\infty} (50) \frac{(-1)^n - 1}{n\pi} \exp\left(-\left(\frac{n\pi}{5}\right)^2 t\right) \sin\left(\frac{n\pi}{5} x\right)$$