Lesson 36

04/15/2022

Finish 9.5, start 9.6.

Last time: heated rod, fixed endpt temperature

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \; t>0 \\ u(0,t) = u(L,t) = 0 & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

length of rod

Separation of variables

Derived:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{L^2}kt} \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

As $t \to \infty$, $u(x,t) \to 0$

The case of insulated endpoints

No heat flowing in & out of endpts

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \; t>0 & (1) \\ u_x(0,t) = u_x(L,t) = 0 & t>0 & (2) \\ u(x,0) = f(x) & 0<x<L & (3) \end{cases}$$

$\to$ known

w/ separation of variables:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}kt} \cos\left(\frac{n\pi}{L}x\right)$$

$u(x,t)$ satisfies (check) (1) & (2)

Want: 3

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) = f(x)$$

satisfied when $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

As $t \to \infty$, $u(x,t) \to \frac{a_0}{2} = \underbrace{\frac{1}{L}\int_0^L f(x)\,dx}_{\text{average of initial temperature.}}$

Ex: $\begin{cases} 5u_t = u_{xx} & 0<x<10,\ t>0 \\ u_x(0,t) = u_x(10,t) = 0 & t>0 \\ u(x,0) = 4x & 0<x<10 \end{cases}$

$k = \frac{1}{5}$, $L = 10$

Find coef. of $f(x) = 4x$

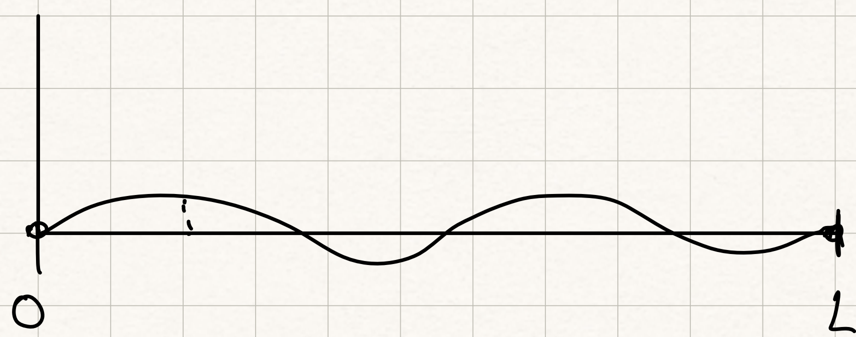
$$a_0 = \frac{2}{10}\int_0^{10} 4x\,dx = \frac{4}{10}x^2\Big|_0^{10} = 40$$

$$a_n = \frac{2}{10}\int_0^{10} 4x\cos\left(\frac{n\pi}{10}x\right)dx = \dots = \frac{80}{n^2\pi^2}\left((-1)^n - 1\right)$$

So:

$$u(x,t) = 20 + \sum_{n=1}^{\infty} \frac{80\left((-1)^n - 1\right)}{n^2\pi^2} e^{-\frac{n^2\pi^2}{10^2}\frac{1}{5}t} \cos\left(\frac{n\pi}{10}x\right)$$

<u>Vibrating string</u>



Model: displacement of a particle of the string from equilibrium.

Assume: particles move only in direction of y axis.

<u>Eq'n:</u> $y_{tt} = a^2 y_{xx}$ $\qquad a^2 = \frac{T}{\rho}$

$T \to$ tension at endpts

$\rho \to$ density of string.

$y(x,t)$ is displacement of particle at x at time t.

Want: endpt & initial conditions.

$$y(0,t) = y(L,t) = 0.$$

To specify a sol'n, need 2 initial conditions

$$\underbrace{y(x,0) = f(x)}_{\text{initial displacement}}, \qquad \underbrace{y_t(x,0) = g(x)}_{\text{initial velocity.}}$$

non-zero initial equlibrium

non zero initial velocity.

Problem:

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0<x<L, \quad t>0 \quad (1) \\ y(0,t) = y(L,t) = 0 & t>0 \quad (2) \\ y(x,0) = f(x) & 0<x<L \quad (3) \\ y_t(x,0) = g(x) & 0<x<L \quad (4) \end{cases}$$

(3), (4) non-homog. conditions $\leftarrow$ issue

1, 2 homog. conditions.

Split problem into 2:

<u>Pr. A:</u>

$$y_{tt} = a^2 y_{xx} \quad (1)$$
$$y(0,t) = y(L,t) = 0 \quad (2)$$
$$y(x,0) = f(x) \quad (3)$$
$$y_t(x,0) = 0 \quad (4)$$

<u>Pr B</u>

$$y_{tt} = a^2 y_{xx}$$
$$y(0,t) = y(L,t) = 0$$
$$y(x,0) = 0$$
$$y_t(x,0) = g(x)$$

Each of A, B has 1 non-homog. cond. and if y_A solves A, y_B solves B then

$$y = y_A + y_B$$

solves original problem.

<u>For Problem A:</u>

Want: $$y(x,t) = \sum_{n=1}^{\infty} c_n \, y_n(x,t)$$

$y_n(x,t) \to$ building blocks satisfying 1, 2, 4

At the end, arrange 3 to be satisfied.

Guess: $y_n(x,t) = X_n(x)\,T_n(t)$

From 1: $X_n T_n'' = a^2 X_n'' T_n$

$$\underbrace{\frac{T_n''}{a^2 T_n}}_{\text{depends on } t} = \underbrace{\frac{X_n''}{X_n}}_{\text{depends on } x} = \underbrace{-\lambda_n}_{\text{const}}$$

$$\Rightarrow \begin{cases} X_n'' + \lambda_n X_n = 0 \\ T_n'' + \lambda_n a^2 T_n = 0 \end{cases}$$

2: $X_n(0)\,T_n(t) = X_n(L)\,T_n(t) = 0$

for all $t \Rightarrow X_n(0) = X_n(L) = 0.$

3: $X_n(x)\,T_n'(0) = 0 \Rightarrow T_n'(0) = 0.$

Start w/ one that has 2 conditions.
There exist non-trivial X_n exactly when $\lambda_n = \left(\frac{n\pi}{L}\right)^2.$