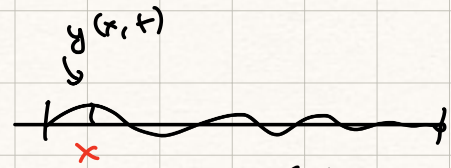
Lesson 37

T: tension
ρ: density
$a = \sqrt{\frac{T}{\rho}}$



Problem:

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0 < x < L,\ t > 0 \quad (1) \\ y(0,t) = y(L,t) = 0 & t > 0 \quad (2) \\ y(x,0) = f(x) & 0 < x < L \quad (3) \\ y_t(x,0) = g(x) & 0 < x < L \quad (4) \end{cases}$$

non homog.

split!

Pr. A:

$$y_{tt} = a^2 y_{xx} \quad (1)$$
$$y(0,t) = y(L,t) = 0 \quad (2)$$
$$y(x,0) = f(x) \quad (3)$$
$$y_t(x,0) = 0 \quad (4)$$

Pr B

$$y_{tt} = a^2 y_{xx}$$
$$y(0,t) = y(L,t) = 0$$
$$y(x,0) = 0$$
$$y_t(x,0) = g(x)$$

Sought soln. for Pr. A:

$$y = \sum_{n=1}^{\infty} A_n\, y_n(x,t)$$

where y_n satisfy (1), (2), (4) and

$$y_n = X_n(x)\, T_n(t)$$

Found: ① $\to \begin{cases} X_n'' + \lambda_n X_n = 0 \\ T_n'' + a^2 \lambda_n T_n = 0 \end{cases}$

② $\to$ $X_n(0) = X_n(L) = 0$

④ $\to$ $\partial_t y(x,0) = 0 \to X_n(x)\, T_n'(0) = 0$

$\Rightarrow T_n'(0) = 0$

① $\begin{cases} X_n'' + \lambda_n X_n = 0 \\ X_n(0) = X_n(L) = 0 \end{cases}$ ② $\begin{cases} T_n'' + a^2 \lambda_n T_n = 0 \\ T_n'(0) = 0 \end{cases}$

Saw (last wed?) ① has non-trivial sol's exactly when $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $X_n = a_n \sin\left(\frac{n\pi}{L} x\right)$

(we can take $a_n = 1$)

Go into ②: $\begin{cases} T_n'' + a^2\left(\frac{n\pi}{L}\right)^2 T_n = 0 \\ T_n'(0) = 0 \end{cases}$

Gen sol'n:

$T_n = A\cos\left(a\frac{n\pi}{L}t\right) + B\sin\left(a\frac{n\pi}{L}t\right)$

$T_n'(0) = 0 \Rightarrow$

π

$$-Aa\frac{n\pi}{L}\sin\left(a\frac{n\pi}{L}t\right)+B\frac{an\pi}{L}\cos\left(a\frac{n}{L}t\right)\Big|_{t=0}=0$$

$\Rightarrow \quad B=0$

$\Rightarrow \quad T_n = A\cos\left(a\frac{n\pi}{L}t\right)$, take $A=1$.

So: soln to Problem A:

$$\boxed{y_A(x,t)=\sum_{n=1}^{\infty}\underbrace{A_n}_{\text{TBD}}\cos\left(a\frac{n\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$$

Pr. A:

$y_{tt}=a^2y_{xx}$ ①

$y(0,t)=y(L,t)=0$ ②

$y(x,0)=f(x)$ ③

$y_t(x,0)=0$ ④

used ①, ②, ④

$$y_A(x,0)=f(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)=\overbrace{f(x)}^{\text{known}}$$

so $\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$ (F. sine series coef of f)

For Problem B:

Pr B

$y_{tt} = a^2 y_{xx}$
$y(0,t) = y(L,t) = 0$
$y(x,0) = 0$
$y_t(x,0) = g(x)$

Exercise: use separation of variables to compute sol'n.

$$\boxed{\begin{aligned} y_B(x,t) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{an\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right) \\ B_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}}$$

not exactly F. sine series coef.

Sol'n to original problem $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ is given by

$y = y_A + y_B$ (check).

Ex: $\begin{cases} 4y_{tt} = y_{xx} & 0 < x < 2, \ t > 0 \\ y(0,t) = y(2,t) = 0 \\ y_t(x,0) = 0 \\ y(x,0) = \frac{1}{5}\sin(\pi x)\cos(\pi x) \end{cases}$

Pr. A:

$y_{tt} = a^2 y_{xx}$ ①

$y(0,t) = y(L,t) = 0$ ②

$y(x,0) = f(x)$ ③

$y_t(x,0) = 0$ ④

Pr B

$y_{tt} = a^2 y_{xx}$

$y(0,t) = y(L,t) = 0$

$y(x,0) = 0$

$y_t(x,0) = g(x)$

$a = \frac{1}{2}$, $L = 2$

So:

$$y = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2} \cdot \frac{1}{2} t\right) \sin\left(\frac{n\pi}{2} x\right)$$

$A_n \rightarrow$ F. sine coef. of $\frac{1}{5}\sin(\pi x)\cos(\pi x)$

$= \frac{1}{10}\sin(2\pi x)$ (double angle)

$$\frac{1}{10}\sin(2\pi x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2} x\right)$$

for some A_n. when $n = 4 \rightarrow \sin(2\pi x)$ on RHS

$A_4 = \frac{1}{10}$

$A_n = 0$ for all $n \neq 4$

$x \in (0,2)$

So: $y(x,t) = \frac{1}{10}\cos(\pi t)\sin(2\pi x)$

(no summation)

//

Vibrating string & sound

Fix x_0:

$$y_n(x_0,t) = \sum_{n=1}^{\infty} \underbrace{A_n}_{\text{const.}} \cos\left(a\frac{n\pi}{L}t\right) \underbrace{\sin\left(\frac{n\pi}{L}x_0\right)}_{\text{const.}}$$

Movement of particle at x_0 → infinite sum of periodic functions of t, w/ frequencies:

$$\frac{1}{2\pi}\frac{a\pi}{L}, \quad \frac{1}{2\pi}\frac{2\pi a}{L}, \quad \frac{1}{2\pi}\frac{3\pi a}{L}, \ldots$$

$$\underbrace{\sqrt{\frac{T}{\rho}}\frac{1}{2L}}, \quad \frac{2}{2L}\sqrt{\frac{T}{\rho}}, \quad \frac{3}{2L}\sqrt{\frac{T}{\rho}}, \ldots$$

ν fundamental freq: lowest.

T → tension, ρ → density.

All other freq. are integer multiples of ν

Freq: do not depend on how the string is plucked, i.e. on the initial conditions.

Freq. depends on tension, length, density.

If $L_2 = \frac{2}{3} L_1$

$$v_2 = \sqrt{\frac{T}{\rho}} \frac{1}{2\left(\frac{2}{3}L\right)} = \frac{3}{2} v_1$$