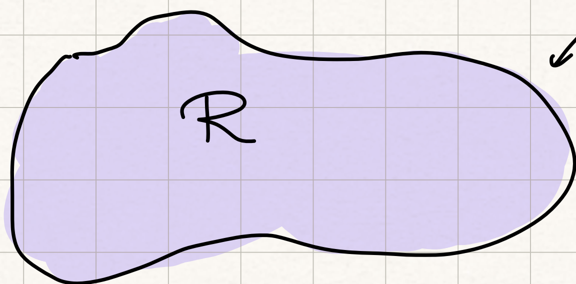
Lesson 38 04/20/22

9.7 Setting: thin 2-dim'l plate (lamina) occupies a domain R in the plane



C : boundary, nice

Faces of lamina are insulated, heat can flow in and out only from boundary

Temperature $\underbrace{u(x,y,t)}$
location in lamina ↳ time

$$\boxed{\frac{\partial u}{\partial t} = k \nabla^2 u}$$ heat eqn in 2 dimensions

k: thermal diffusivity

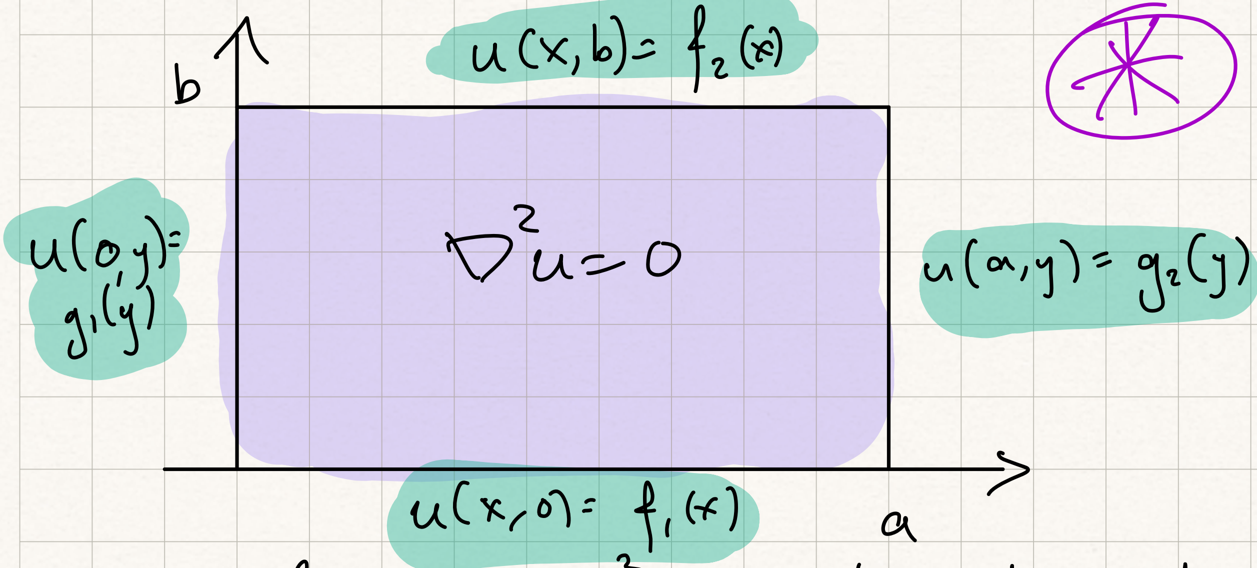
∇^2 : Laplace operator / Laplacian

$$\nabla^2 u = \partial_x^2 u + \partial_y^2 u$$ (other notation: Δ)

Restrict interest in u const. in time.

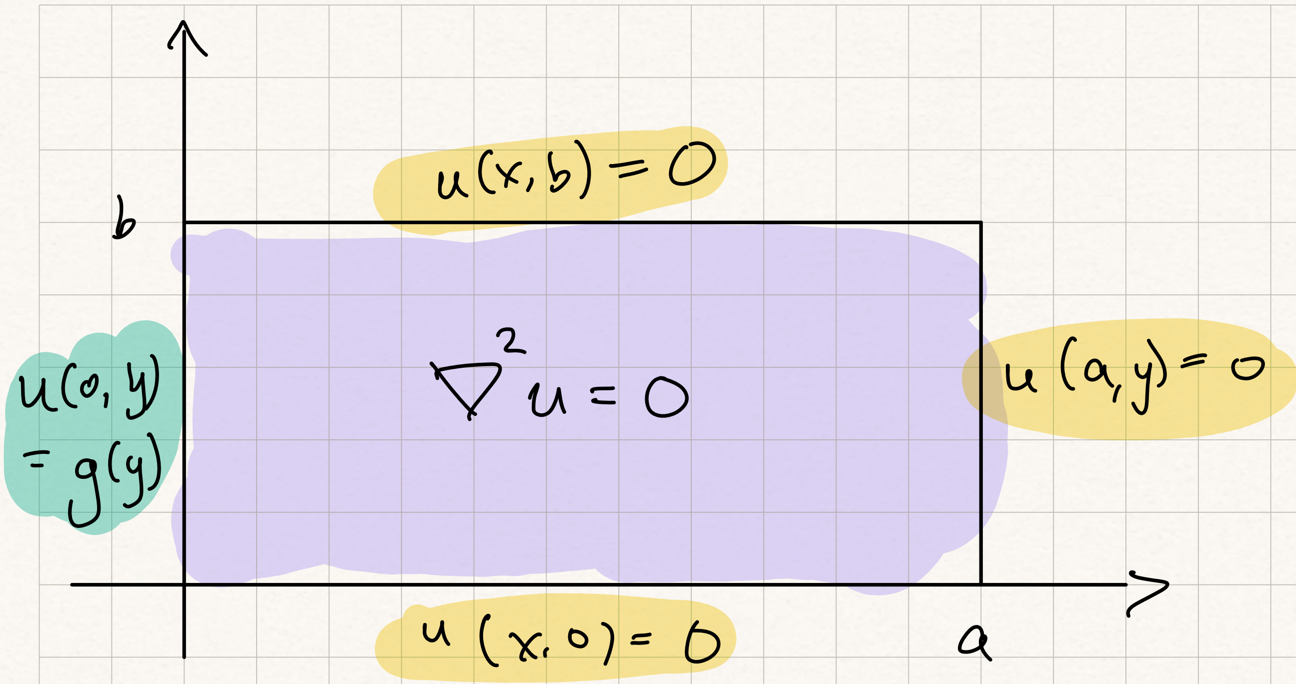
$$\boxed{\nabla^2 u = 0}$$ Laplace eqn.

Domain : Rectangle for today *



Terminology: $\nabla^2 u = 0$ w/ values of u prescribed at boundary is called a Dirichlet problem. Has unique sol'n for nice boundary and boundary conditions.

* : 4 non-homog. cond, split into sub-problems, each w/ only one non-hom. cond.



Seek sol's: $u(x,y) = \sum_{n=1}^{\infty} c_n \, u_n(x,y)$

$u_n \to$ building blocks

$$\begin{cases} \nabla^2 u_n = 0 & (1) \\ u_n(x,b) = 0 & (2) \\ u_n(a,y) = 0 & (3) \\ u_n(x,0) = 0 & (4) \end{cases}$$

Assume: $u_n(x,y) = X_n(x)\, Y_n(y)$

(1) Plug into $\partial_x^2 u_n + \partial_y^2 u_n = 0$

$\to$ find 2 ODEs for X_n, Y_n

$$\partial_x^2 u_n + \partial_y^2 u_n = 0 \Rightarrow$$

$$X_n''(x) Y_n(y) + X_n(x) Y_n''(y) = 0$$

$$\Rightarrow -\frac{X_n''}{X_n}(x) = \frac{Y_n''}{Y_n}(y) = -\lambda_n$$

$$\Rightarrow \begin{cases} X_n'' - \lambda_n X_n = 0 \\ Y_n'' + \lambda_n Y_n = 0 \end{cases}$$

(2) Plug u_n into (2) - (4):

(2) $\Rightarrow X_n(x)\, Y_n(b) = 0 \xRightarrow{X_n(x) \not\equiv 0} Y_n(b) = 0$

(3) $\Rightarrow X_n(a)\, Y_n(y) = 0 \Rightarrow X_n(a) = 0$

(4) $\Rightarrow X_n(x)\, Y_n(0) = 0 \Rightarrow Y_n(0) = 0$

I $\begin{cases} Y_n'' + \lambda_n Y_n = 0 \\ Y_n(0) = 0, \quad Y_n(b) = 0 \end{cases}$

II $\begin{cases} X_n'' - \lambda_n X_n = 0 \\ X_n(a) = 0 \end{cases}$

wanted (+) in the problem w/ 2 endpt cond.

(3) Solve problem w/ $\underline{\underline{2}}$ endpt conditions

→ find λ_n

I : $\lambda_n = \left(\frac{n\pi}{b}\right)^2$

and $Y_n(y) = A \sin\left(\frac{n\pi}{b} y\right)$ (can take $A = 1$)

④ Solve problem $\textcircled{\text{II}}$

$$\begin{cases} X_n'' - \left(\frac{n\pi}{b}\right)^2 X_n = 0 \\ X_n(a) = 0 \end{cases}$$

$$X_n(x) = A_n e^{\frac{n\pi}{b}x} + B_n e^{-\frac{n\pi}{b}x}$$

(Can also write $X_n(x) = \tilde{A}_n \cosh\left(\frac{n\pi}{b}x\right) + \tilde{B}_n \sin\left(\frac{n\pi}{b}x\right)$)

$X_n(a) = 0$

$$\Rightarrow A_n e^{\frac{n\pi}{b}a} + B_n e^{-\frac{n\pi}{b}a} = 0$$

$$\Rightarrow A_n = -B_n e^{-\frac{2n\pi}{b}a}$$

$$\Rightarrow X_n(x) = -B_n e^{-\frac{2n\pi}{b}a} e^{\frac{n\pi x}{b}} + B_n e^{-\frac{n\pi}{b}x}$$

(optional)

$$= -B_n e^{-\frac{n\pi}{b}a} e^{-\frac{n\pi}{b}a} e^{\frac{n\pi x}{b}} + B_n e^{-\frac{n\pi}{b}x}$$

$$= B_n e^{-\frac{n\pi}{b}a}\Big(\underbrace{-e^{-\frac{n\pi}{b}a} e^{\frac{n\pi x}{b}} + e^{\frac{n\pi}{b}a} e^{-\frac{n\pi}{b}x}}\Big)$$

$$= 2B_n e^{-\frac{n\pi}{b}a} \sinh\left(\frac{n\pi}{b}(a-x)\right)$$

→ ignore (constant)

$$X_n(x) = \sinh\left(\frac{n\pi}{b}(a-x)\right)$$

5 Write:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \underset{\uparrow \atop \text{TBD}}{c_n} \sinh\left(\frac{n\pi}{b}(a-x)\right)\sin\left(\frac{n\pi}{b}y\right)}$$

6 Use non-homog. condition to find c_n:

$$u(0,y) = g(y)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right) = \underset{\text{known}}{g(y)}$$

Take: $\quad c_n \sinh\left(\frac{n\pi}{b}a\right) = \frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi}{b}y\right)dy$

$$\Rightarrow \quad \boxed{c_n = \frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b g(y)\sin\left(\frac{n\pi}{b}y\right)dy}$$