Lesson 4. 01/19/2022

$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}}$ $n \times n$ const. matrix

If λ is an eigenvalue of $\underline{\underline{A}}$ $(\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0)$ w/ associated eigenvector $\underline{\underline{v}}$ $((\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{\underline{v}} = \underline{\underline{0}})$ then $\underline{\underline{x}}(t) = e^{\lambda t}\underline{\underline{v}}$ is a sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$.

Method: Given $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ as before:

1. Solve characteristic eq'n
$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$
to find eigenvalues $\lambda_1, \dots, \lambda_n$.
(n eigenvalues, possibly repeated or complex)
2. Find associated eigenvectors $\underline{\underline{v}}_1, \dots, \underline{\underline{v}}_n$
3. If process yields n linearly independent e-vectors then:
$$\underline{\underline{x}}_1(t) = e^{\lambda_1 t}\underline{\underline{v}}_1, \dots, \underline{\underline{x}}_n(t) = e^{\lambda_n t}\underline{\underline{v}}_n$$
are n lin. independent sol's of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$.
4. Any sol'n is of the form
$$\underline{\underline{x}} = c_1 \underline{\underline{x}}_1(t) + \dots + c_n \underline{\underline{x}}_n(t).$$

Fact: If $\lambda_1, \dots, \lambda_n$ are distinct (all different from each other) then Step 3 works.

Ex:

$$\underline{A} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}, \quad \underline{x}' = \underline{A}\,\underline{x}$$

1. Find e-values.

$$\det \begin{bmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{bmatrix} =$$

$$= (5-\lambda) \begin{vmatrix} -1-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} - 6 \begin{vmatrix} 2 & -1-\lambda \\ 4 & -2 \end{vmatrix}$$

$$= (5-\lambda)\Big((-1-\lambda)(-4-\lambda)-4\Big) - 6\Big(-4-4(-1-\lambda)\Big)$$

$$= \dots = \lambda - \lambda^3 = \lambda(1-\lambda^2)$$

$$\lambda(1-\lambda^2) = 0 \dashrightarrow \underbrace{\lambda = 0, \ \lambda = 1, \ \lambda = -1}$$

distinct e-values, method works!

2. Find e-vectors.

i) $\lambda = 0$

$$(A - 0 I)v = 0$$

$$\Rightarrow \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 5v_1 - 6v_3 = 0 \\ 2v_1 - v_2 - 2v_3 = 0 \\ \cancel{4v_1 - 2v_2 - 4v_3 = 0} \end{cases}$$

multiple of 2nd line.

$$\Rightarrow \begin{cases} v_1 = \frac{6}{5}v_3 \\ v_2 = 2 \cdot \frac{6}{5}v_3 - 2v_3 = \frac{2}{5}v_3 \end{cases}$$

An e-vector is: $v = \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$

Sos a sol'n to $x' = Ax$ is

$$x_1(t) = e^{0t} \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

$\underline{\lambda = 1}$ $\quad (\underline{\underline{A}} - \underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} 1. \\ 2. \\ 3. \end{array} \left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 2 & -2 & -2 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

Elementary row operations.

1. Interchange rows
2. Multiply row by non-zero #
3. Add multiple of a row to another row.

→ equivalent system.

__Goal:__ 1 at top left, 0 under it.

$② \cdot \frac{1}{2} \to ②$

$$\left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 1 & -1 & -1 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$② \leftrightarrow ①$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 0 & -6 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$(2) - 4(1) \to (2)$

$(3) - 4\cdot(1) \to (3)$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$(3) - \frac{1}{2}(2) \to (3)$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} v_1 - v_2 - v_3 = 0 \\ 4v_2 - 2v_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} v_3 = 2v_2 \\ v_1 = 3v_2 \end{cases}$$

An eigenvector: $\underline{\underline{v}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

A soln of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is

$$\underline{\underline{x}}_2(t) = e^{t} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Exercise: find an e.vector $\underline{\underline{v}}_3$ associated to $\lambda_3 = -1$

so: gen. sol'n:

$$\underline{\underline{x}}(t) = c_1 \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-t}\, \underline{\underline{v}}_3$$ //

What if we have cplx eigenvalues?

Complex Numbers

Numbers of the form $z = a + ib$

$a, b \in \mathbb{R}$, $i^2 = -1$

Ex: $2 + 3i$

Denote set of cplx numbers by $\mathbb{C}$.

Real part: $\quad Re(z) = a$

Imaginary part: $\quad Im(z) = b$ ← Imaginary part is a real number!

Ex: $\quad Re(2+3i) = 2, \quad Im(2+3i) = 3$

Complex conjugate: $\quad z = a+bi, \quad \bar{z} = a-bi$

Note: $\quad |z|^2 = z \cdot \bar{z} = (a+bi)(a-bi)$

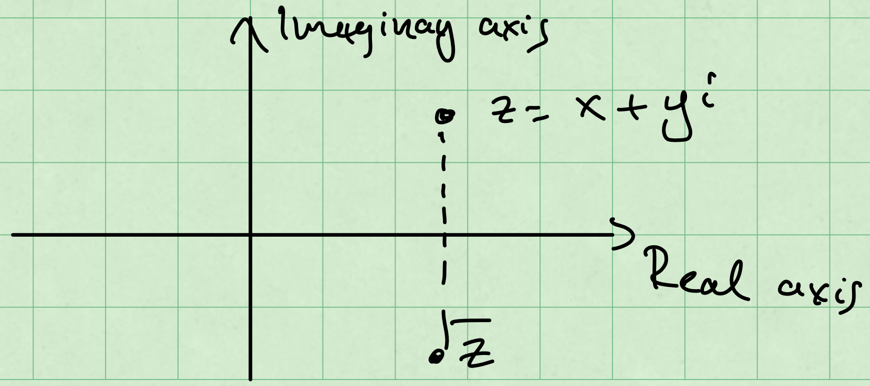
$$= a^2 - (bi)^2$$

$$= a^2 + b^2$$ ← always real.

Invert cplx numbers $z \neq 0$

$$\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

Ex: $\quad \frac{1}{2+3i} = \frac{2}{4+9} - \frac{3}{4+9} i = \frac{2}{13} - \frac{3}{13} i$



Ex for next time:

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}, \quad \underline{\underline{A}} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$$

Find e-values:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -3-\lambda & 4 \\ -4 & -3-\lambda \end{vmatrix}$$

$$\Rightarrow (-3-\lambda)^2 + 16 = 0$$

$$\Rightarrow -3-\lambda = \pm i\sqrt{16}$$

$$\Rightarrow \lambda = -3 \pm 4i$$

Eigenvalues: $\lambda_1 = -3+4i$
$\lambda_2 = -3-4i$

are conjugates of each other (always true for matrices w/ real entries).

Eigenvectors on Friday.