Lesson 40

04/25/2022

Final → cumulative, Thursday of finals week.

10.1 Sturm-Liouville Problems

Seen: $\begin{cases} x'' + 4x = f(t) & (1) \\ x(0) = x(L) = 0 \end{cases}$

Expanded $f(x)$ into infinite sum of building blocks:

$$f(t) = \sum_{n=1}^{\infty} b_n \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{y_n(x)}$$

found: $x(t) = \sum_{n=1}^{\infty} B_n y_n(t)$ (uses that (1) has no x' term)

$$x'' + 4x = f(t)$$

$$\Rightarrow \sum_{n=1}^{\infty} B_n y_n'' + 4 \sum_{n=1}^{\infty} B_n y_n = \sum_{n=1}^{\infty} b_n y_n$$

Nice property #1: $y_n'' = -\left(\frac{n\pi}{L}\right)^2 y_n$

$$\sum_{n=1}^{\infty} B_n\left(-\left(\frac{n\pi}{L}\right)^2\right) y_n + 4 \sum_{n=1}^{\infty} B_n y_n = \sum_{n=1}^{\infty} b_n y_n$$

Matching coef: $B_n\left(-\left(\frac{n\pi}{L}\right)^2 + 4\right) = \overbrace{b_n}^{\text{known}}$

$$B_n = \frac{1}{4 - \left(\frac{n\pi}{L}\right)^2} b_n$$

Nice property #2: $y_n(0) = y_n(L) = 0$

$$\Rightarrow \quad x(t) = \sum_{n=1}^{\infty} B_n \, y_n(t)$$

then $x(0) = x(L) = 0.$

Our building blocks were good bec. they satisfied

$$(*)\begin{cases} y_n'' + \lambda_n y_n = 0 \\ y_n(0) = y_n(L) = 0 \end{cases} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Sturm - Liouville Problems (generalization of $(*)$. Can help us solve more general versions of (1).

$$(2)\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) - q(x)\,y + \lambda\, r(x)\,y = 0 \\ \qquad a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

Neither α_1 & α_2 both 0 nor β_1 & β_2 both 0.

$p(x)$, $q(x)$, $r(x)$ $\rightarrow$ known nice functions.

In this class: $p \equiv 1$, $q \equiv 0$, $r \equiv 1$

$$y'' + \lambda y = 0.$$

Notice: (2) is always solved by $y \equiv 0$.

λ: tbd so that (2) has non-trivial sol's (i.e. sol's that are not $\equiv 0$).

Such λ is called an <u>eigenvalue</u>, cor. non-trivial sol'n is called an <u>eigenfunction</u>

<u>Ex 1:</u>
$$\begin{cases} y'' + \lambda y = 0 & (1) \\ y(0) = y(L) = 0 & (2) \end{cases}$$

(1) $\rightarrow$ special case of S-L eq'n.

(2)
$$\begin{cases} 1\, y(0) - 0\, y'(0) = 0 \\ 1 \cdot y(L) + 0\, y'(L) = 0 \end{cases}$$

(1) + (2) is a S-L problem.

<u>Found:</u> it has non-trivial sol'n exactly when $\lambda = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, \ldots$

$$y_n = B \sin\left(\frac{n\pi}{L} x\right)$$

for $B \in \mathbb{R}$.

Eigenvalues: $\lambda = \left(\frac{n\pi}{L}\right)^2$ w/ eigenfcts $y_n = B \sin\left(\frac{n\pi}{L} x\right)$ //

Ex 2: $\begin{cases} y'' + \lambda y = 0 \quad (*) \\ y(0) = 0, \quad y'(L) = 0 \end{cases}$ $\qquad 0 < x < L$

$\begin{cases} 1\,y(0) - 0\,y'(0) = 0 \\ 0\,y(L) + 1\,y'(L) = 0 \end{cases}$

$\alpha_1 y(a) - \alpha_2 y'(a) = 0$
$\beta_1 y(b) + \beta_2 y'(b) = 0$

So this is a S-L problem.

Try to find e-values:

$\lambda = 0$: Find general of $(*)$, plug into endpt conditions to look for non-trivial sol's.

$y(x) = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y'(x) = A \Rightarrow \left(y'(L) = 0 \Rightarrow A = 0 \right)$

so $\lambda = 0$ is not an eigenvalue.

$\lambda = -\alpha^2$ Gen. sol'n:

$y = Ae^{\alpha x} + Be^{-\alpha x}$

$y(0) = 0 \Rightarrow A + B = 0$

$y'(L) = 0 \Rightarrow \alpha A e^{\alpha L} - \alpha B e^{-\alpha L} = 0$

$\Rightarrow \alpha A e^{\alpha L} + \alpha A e^{-\alpha L} = 0$

$\Rightarrow \underbrace{\alpha}_{\neq 0} A \underbrace{(e^{\alpha L} + e^{-\alpha L})}_{> 0} = 0$

$\Rightarrow A = 0 \Rightarrow B = 0$

No negative eigenvalues.

$\underline{\lambda = \alpha^2}$ $\qquad y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$$y(0) = 0 \Rightarrow A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$$

$$y(x) = B\sin(\alpha x)$$

$$y'(L) = 0 \Rightarrow \alpha B\cos(\alpha L) = 0$$

Want: $B \neq 0$, $\alpha \neq 0$, tbd.

$L > 0$ known.

Non-zero B exists exactly when $\cos(\alpha L) = 0$

$$\Rightarrow \alpha L = (2n-1)\frac{\pi}{2} \qquad n = 1, 2, 3, \ldots$$

(odd multiple of $\frac{\pi}{2}$)

$$\Rightarrow \alpha = \frac{(2n-1)}{L}\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

Eigenvalues: $\lambda = \left(\frac{2n-1}{L}\frac{\pi}{2}\right)^2$, w/

eigenfunctions:

$$y_n(x) = B\sin\left(\frac{2n-1}{L}\frac{\pi}{2}x\right), \quad n = 1, 2, 3, \ldots //$$

Note: The endpt conditions of a SL problem affect e-values & e-fcts: Ex 1 & Ex 2: $y'' + \lambda y = 0$, but different endpt conditions led to different e-values & e-fcts.