Lesson 41 04/27/22

Last time:

Sturm - Liouville Problems

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x) y = 0 \\ \qquad\qquad\qquad\qquad a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

Ex 1: $y'' + \lambda y = 0$

$y(0) = y(L) = 0$

E-values: $\lambda = \left(\frac{n\pi}{L}\right)^2$, E-fcts $y_n = \sin\left(\frac{n\pi}{L}x\right)$

Ex 2: $y'' + \lambda y = 0$

$y(0) = y'(L) = 0$

$$\lambda = \left(\frac{(2n-1)}{L}\frac{\pi}{2}\right)^2, \quad \text{E-fcts: } y_n = \sin\left(\frac{2n-1}{L}\frac{\pi}{2}x\right)$$

$n = 1, 2, \ldots$

Thm: If p, q, r nice, $p > 0, r > 0$ then e-values are an infinite increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

$$\lim_{n\to\infty} \lambda_n \to \infty$$

If $q \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$

$\Rightarrow \lambda_j \geq 0$

A S-L problem satisfying those properties is called regular.

Ex 3 $\quad y'' + \lambda y = 0 \qquad 0 < x < L$

$hy(0) - y'(0) = 0 \quad , y(L) = 0 \qquad h > 0$ given

$L > 0$ given

$hy(0) - 1 \cdot y'(0) = 0$

$1 \cdot y(L) - 0 \cdot y'(L) = 0$

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

$\Rightarrow$ regular S-L problem

$\Rightarrow \quad \lambda_j \geq 0$

Find E-values:

$\lambda = 0 \qquad y'' = 0 \Rightarrow y(x) = Ax + B$

$hy(0) - y'(0) = 0 \Rightarrow hB - A = 0$

$\Rightarrow A = hB$

$y(L) = 0 \Rightarrow AL + B = 0 \Rightarrow$

$\underbrace{(hL + 1)}_{> 0} B = 0$

$\Rightarrow B = 0$

$A = 0$

$\lambda = 0$ is <u>not</u> an e-value.

<u>$\lambda = \alpha^2$</u> $\quad y'' + \alpha^2 y = 0$

$$y(x) = A\cos(\alpha x) + B\sin(\alpha x)$$
$$y'(x) = -A\alpha\sin(\alpha x) + B\alpha\cos(\alpha x)$$

<u>So:</u>

$$hy(0) - y'(0) = 0 \Rightarrow$$

$$hA - B\alpha = 0 \Rightarrow A = \frac{\alpha B}{h}$$

$$y(L) = 0 \Rightarrow A\cos(\alpha L) + B\sin(\alpha L) = 0$$

$$\frac{\alpha B}{h}\cos(\alpha L) + B\sin(\alpha L) = 0$$

$L, h \rightarrow$ known

$\alpha \rightarrow$ tbd so that $B \neq 0$

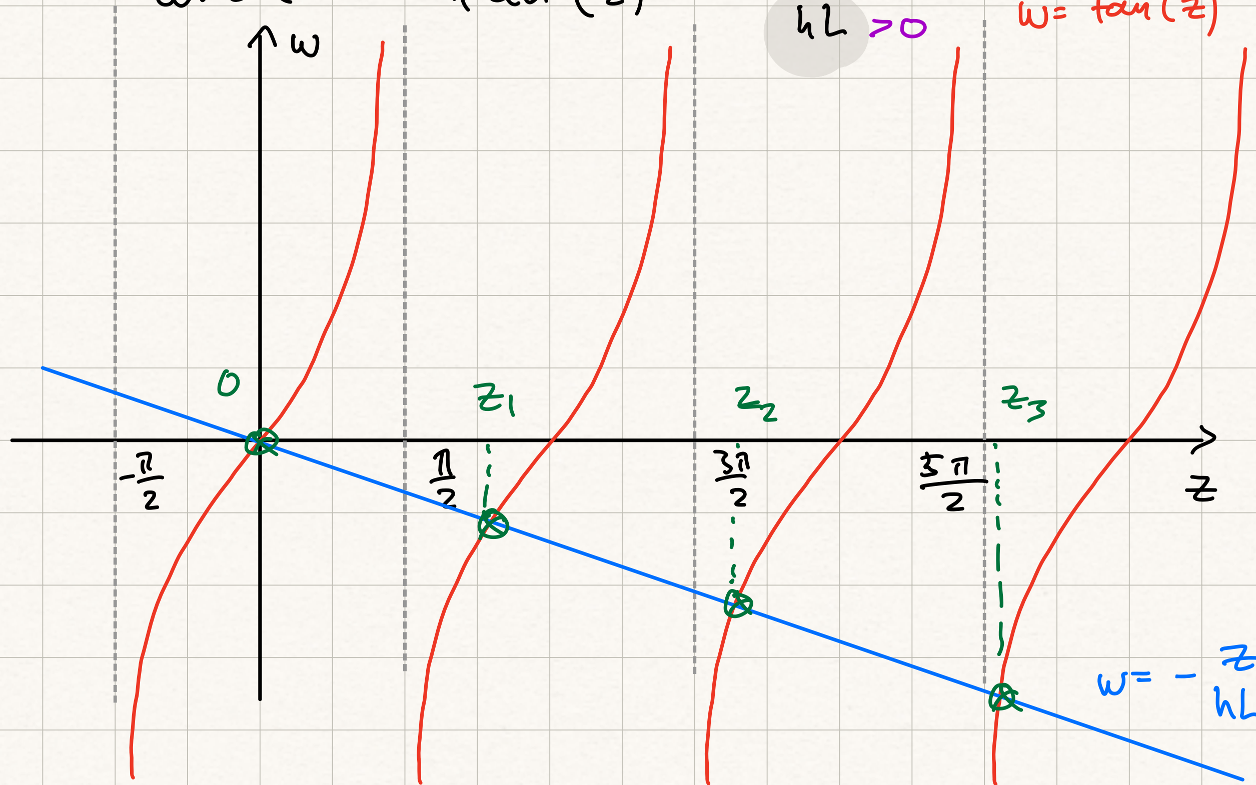
want to solve for α in

$$\frac{\alpha}{h}\cos(\alpha L) + \sin(\alpha L) = 0$$

$$\Rightarrow \frac{\alpha}{h}\cos(\alpha L) = -\sin(\alpha L)$$

$$\Rightarrow \tan(\alpha L) = -\frac{\alpha L}{hL} \quad (*)$$

$\Rightarrow$ α works exactly when $\alpha L = z$,
where $\tan(z) = -\frac{z}{hL}$ >0



Sol's of $\tan(z) = -\frac{z}{hL}$ are the z-coordinates of the intersection pts of the 2 graphs.

<u>So</u>: α solves $(*)$ exactly when $\alpha = \frac{z}{L}$, z is one of those sol's above.

E-values: $\lambda_n = \alpha_n^2 = \left(\frac{z_n}{L}\right)^2$, $z_n \to n$-th positive sol'n of $\tan(z) = -\frac{z}{hL}$

<u>E-fcts:</u>

$$y_n(x) = \frac{\alpha_n B}{h}\cos(\alpha_n x) + B\sin(\alpha_n x),$$ for any $B \in \mathbb{R}$. //

Recall: F. sine series of $f(x)$ on $[0,L]$

$$f(x) = \sum_{n=1}^{\infty} b_n \underbrace{\sin\left(\frac{n\pi}{L}x\right)}$$

ε-fcts of SL-problem

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(L) = 0 \end{cases}$$

Works for more general S-L.

Fact I: The ε-fcts of a SL problem as in beginning of class are orthogonal if they cor. to different e-values

$$\int_a^b y_i(x)\, y_j(x)\, \underbrace{r(x)}_{=1 \text{ for } y''+\lambda y=0}\, dx = 0 \text{ if } i \neq j.$$

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x) y = 0 \quad a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

Fact II: E-fcts of regular S-L problems can be used as building blocks:

$$(*)\quad f(x) = \sum_{n=1}^{\infty} c_n \underbrace{y_n(x)}_{\text{e-fct cor. } \lambda_n} \quad \text{on } [a,b]$$

$$c_n = \frac{1}{\int_a^b (y_n(x))^2 \underbrace{r(x)}_{=1} dx} \int_a^b f(x) y_n(x) \overbrace{r(x)}^{=1} dx$$

(*)

Ex: F. sine series coef:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

e-fct of
$y'' + \lambda y = 0$
$y(0) = y(L) = 0$,

Check:

$$(*)\quad \int_0^L \left(\sin\left(\frac{n\pi}{L} x\right)\right)^2 dx = \frac{L}{2}$$

If f piecewise smooth, (*) converges to

$\to f(x)$ if f cont. at x

$\to \frac{1}{2}\left(f(x^-) + f(x^+)\right)$ if not.

Ex: (*) $\begin{cases} y'' + \lambda y = 0 \\ y(0) = y'(1) = 0 \end{cases}$

Expand $f(x)=3$ on $[0,1]$ in terms of e-fcts of (*)

Seen: $y_n = \sin\left(\frac{2n-1}{2}\pi x\right)$ $n=1,2,\dots$

By (*)

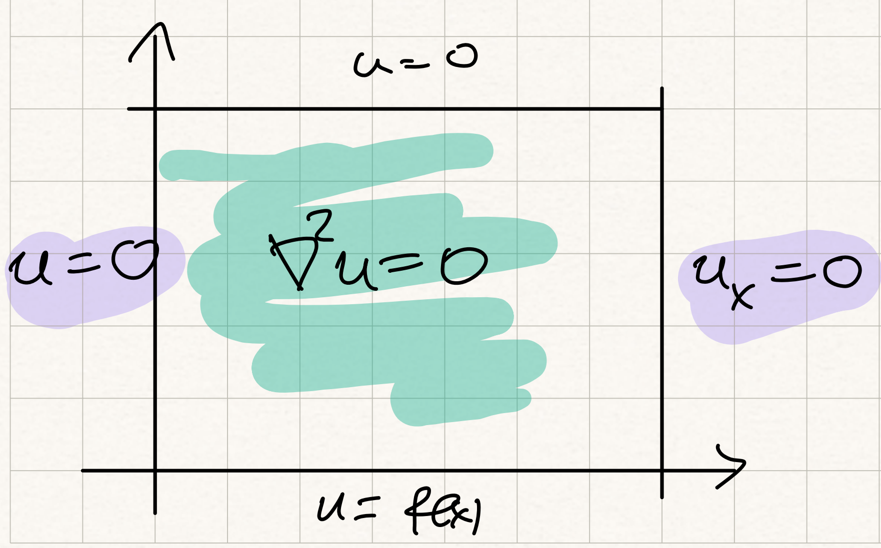
use double angle

$$c_n = \frac{\overbrace{\int_0^1 3 \sin\left(\frac{2n-1}{2}\pi x\right)dx}^{=\frac{6}{\pi(2n-1)}}}{\underbrace{\int_0^1\left(\sin\left(\frac{2n-1}{2}\pi x\right)\right)^2 dx}_{\frac{1}{2}=}}$$

$$c_n = \frac{12}{\pi(2n-1)}$$

So $$3 = \sum_{n=1}^{\infty} \frac{12}{\pi(2n-1)} \sin\left(\frac{2n-1}{2}\pi x\right)$$ //

Application



Sep. of variables $u(x,t) = X(x)\,T(t)$

$$X'' + \lambda X = 0$$

$$X(0) =$$