10 (2009)   Sep. of variables

$$x' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = A$$

$\lambda = 3 \to$ mult. 2 $\to$ defective

$\lambda = 1 \to$ mult 1

$\xi = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is an e.vector for $\lambda = 3$

Sol'n: $x(t) = te^{3t}\xi + e^{3t}\eta$, $\xi = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

gen. e-vector of rank 2

$$\boxed{\begin{aligned} (A - 3I)^2 \eta &= 0 \\ (A - 3I)\eta &= \xi \end{aligned}}$$

$\boxed{\eta}$ e-vector for $\lambda = 1$

$\boxed{\xi} - \boxed{\xi}$

true e-vector

$$A - 3I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A - 3I)^2 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If $\quad u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$:

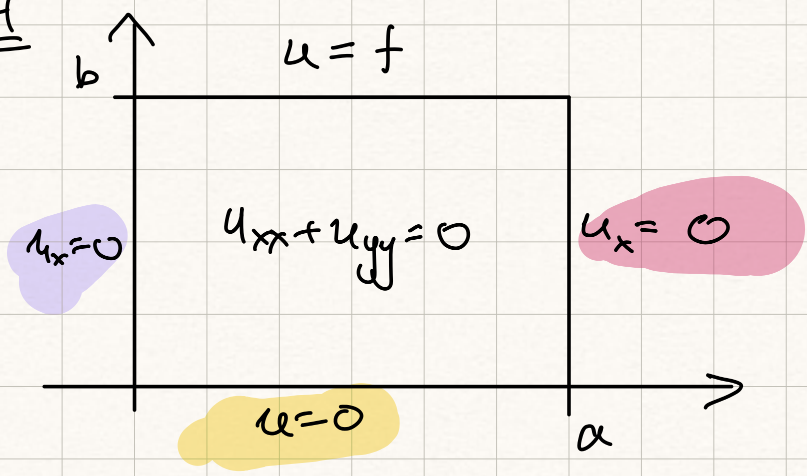
$$4a - 2b + c = 0$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$-2a + b = 1$$
$$c = 2$$

$$B = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 \\ -11 \\ 2 \end{pmatrix}$$

4



$$u(x,y) = \sum_{n \geq 0} c_n \, X_n(x) \, Y_n(y)$$

$$u_n(x,y) = X_n(x) \, Y_n(y)$$

$$\partial_x^2 u_n + \partial_y^2 u_n \Rightarrow \quad X_n'' Y_n + X_n Y_n'' = 0$$

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ Y_n'' - \lambda Y_n = 0 \end{cases}$$

$$X_n'(0) = 0, \quad X_n'(a) = 0$$

$$Y_n(0) = 0$$

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n'(0) = 0, \quad X_n'(a) = 0 \end{cases}$$

$\lambda = 0$

$X_0'' = 0 \Rightarrow \quad X_n(x) = Ax + B$

$X_0'(0) = X_0'(a) = 0 \quad \Rightarrow \quad A = 0$

$B \to$ anything

$\lambda = 0 \to$ eigenvalue

$X_0(x) = B$ eigenfunction.

Building block for $n=0$:

$X_0(x)\, Y_0(y)$

where

- $Y_0'' + 0 \cdot Y_0 = 0$

$Y_0(0) = 0$

$Y_0(y) = Ay + B \quad \Rightarrow B = 0$ by

So: $\quad u_0(x, y) = By$

$\quad \quad \quad \quad B = 1$ take it to be 1

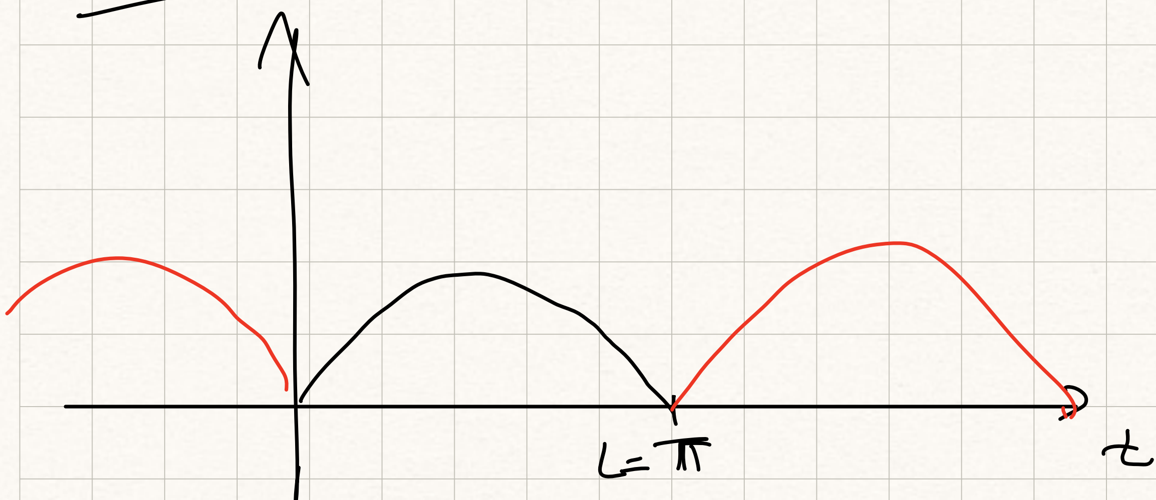
$\lambda > 0$

$\lambda = \left(\frac{n\pi}{a}\right)^2 \quad X_n(x) = \cos\left(\frac{n\pi}{a}x\right)$

$Y_n(y) = \sinh\left(\frac{n\pi}{a}y\right)$

$$u(x,y) = c_0 y + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

9.3   9.



Cosine series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cancel{\pi}}{\cancel{\pi}} t\right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt$$

1. 2 IBP
2. $\sin(t) = \dfrac{e^{it}-e^{-it}}{2i}$, $\cos(nt) = \dfrac{e^{int}+e^{-int}}{2}$

take real part at the end.

3. Use identity

$$\sin(a)\cos(b) = \frac{1}{2}\left(\cos(b-a)+\cos(b+a)\right)$$

$$I = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt$$

$$= -\frac{2}{\pi}\int_0^{\pi} \left(\cos(t)\right)' \cos(nt)\,dt$$

$$= -\frac{2}{\pi}\cos(t)\cos(nt)\Big|_0^{\pi} - \frac{2}{\pi}n\int_0^{\pi}\cos(t)\sin(nt)\,dt$$

$$= -\frac{2}{\pi}\left(-1\left(\cos(n\pi)\right)-1\right) - \frac{2n}{\pi}\int_0^{\pi}\left(\sin(t)\right)'\sin(nt)\,dt$$

$$= -\frac{2}{\pi}\left(-(-1)^n-1\right) - \frac{2}{\pi n}\left(\sin(t)\sin(nt)\right)\Big|_0^{\pi} \rightarrow 0$$

$$+\frac{2}{\pi}n^2\int_0^{\pi}\sin(t)\cos(nt)\,dt$$

$\rightarrow = I$

$$I(1-n^2) = -\frac{2}{\pi}\left(-(-1)^n - 1\right)$$

$$I = -\frac{2}{\pi(1-n^2)}\left(-(-1)^n - 1\right.$$