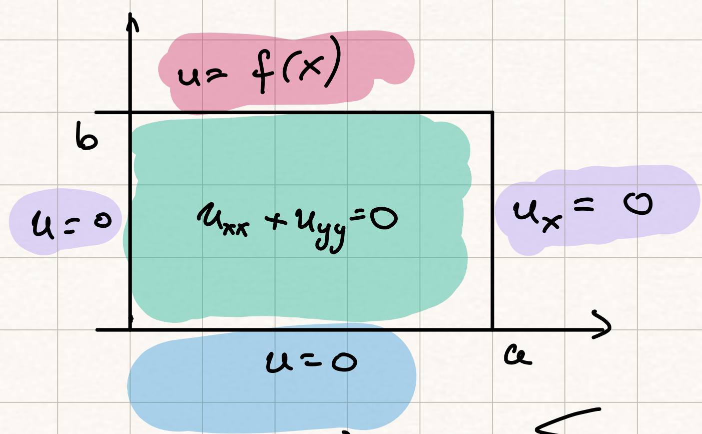
Dirichlet Problem



$$u(x,y) = \sum c_n u_n(x,y)$$

$$u_n(x,y) = X_n(x)\, Y_n(y)$$

$$\partial_x^2 u_n + \partial_y^2 u_n = 0 \Rightarrow X_n'' Y_n + X_n Y_n'' = 0$$

$$\Rightarrow \frac{X_n''}{X_n} = -\frac{Y_n''}{Y_n} = -\lambda$$

$$\Rightarrow \begin{cases} X_n'' + \lambda X_n = 0 \\ Y_n'' - \lambda Y_n = 0 \end{cases}$$

$$u_n(0,y) = 0 \Rightarrow X_n(0)\, Y_n(y) = 0$$

$$\Rightarrow X_n(0) = 0$$

$$\partial_x u_n(a,y) = 0 \Rightarrow X_n'(a)\, Y_n(y) = 0$$

$$\Rightarrow X_n'(a) = 0$$

$$u_n(x,0) = 0 \Rightarrow Y_n(0) = 0$$

(1) $\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n(0) = X_n'(a) = 0 \end{cases}$

(2) $\begin{cases} Y_n'' - \lambda Y_n = 0 \\ Y_n(0) = 0 \end{cases}$

(1): S-L problem. Solved on Monday

E-values $\lambda_n = \left(\frac{(2n-1)\pi}{2a}\right)^2$ $n = 1, 2, \ldots$

E-fcts: $X_n = \sin\left(\frac{(2n-1)\pi}{2a}x\right)$

How this was derived: $\lambda \geq 0$ for regular S-L problems

$\lambda = 0$: $X_n'' = 0 \rightarrow X_n = Ax + B$

$X_n(0) = 0 \Rightarrow B = 0$

$X_n'(a) = 0 \Rightarrow A = 0$

so $\lambda = 0$ not an e-value.

$\lambda = \alpha^2$:

$$X_n'' + \alpha^2 X_n = 0$$

$$X_n = A\cos(\alpha x) + B\sin(\alpha x)$$

$X_n(0) = 0 \Rightarrow A = 0$

$$X_n(x) = B\sin(\alpha x)$$

$$X_n'(a) = 0 \Rightarrow B\alpha\cos(\alpha a) = 0$$

$$\alpha a = (2n-1)\frac{\pi}{2} \qquad n = 1, 2, \ldots$$

$$\Rightarrow \alpha = \frac{2n-1}{2a}\pi$$

$$\Rightarrow \lambda = \left(\frac{2n-1}{2a}\pi\right)^2$$

$$u_n = X_n(x)\, Y_n(y)$$

$$(2) \Rightarrow \begin{cases} Y_n'' - \left(\frac{2n-1}{2a}\pi\right)^2 Y_n = 0 \\ Y_n(0) = 0 \end{cases}$$

$$Y_n(y) = A\cosh\left(\frac{2n-1}{2a}\pi y\right) + B\sinh\left(\frac{2n-1}{2a}\pi y\right)$$

$$\left(\text{Can write } \tilde{A}e^{\frac{2n-1}{2a}\pi y} + \tilde{B}e^{-\frac{2n-1}{2a}y}\right)$$

$$Y_n(0) = 0 \Rightarrow A \cdot 1 + B \cdot 0 = 0$$

$$\Rightarrow A = 0$$

$$Y_n(y) = B\sinh\left(\frac{2n-1}{2a}\pi y\right)$$

(set $B=1$)

$$\underbrace{\boxed{u_n(x,y) = \sin\left(\frac{2n-1}{2a}\pi x\right)\sinh\left(\frac{2n-1}{2a}\pi y\right)}}_{\text{building blocks}} \quad n=1,2,\ldots$$

Sol'n:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{2n-1}{2a}\pi x\right)\sinh\left(\frac{2n-1}{2a}\pi y\right)$$

Want $u(x,b) = f(x)$

$$u(x,b) = \sum_{n=1}^{\infty} \overbrace{c_n \sinh\left(\frac{2n-1}{2a}\pi b\right)}^{\text{constant,}} \sin\left(\frac{2n-1}{2a}\pi x\right)$$

$\shortparallel$

$f(x) \to$ want an expansion for f in terms of e-fcts. to SL problem (I)

Write:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n-1}{2a}\pi x\right)$$

$$B_n(x) = \frac{\int_0^a f(x) \sin\left(\frac{2n-1}{2a}\pi x\right) dx}{\int_0^a \left(\sin\left(\frac{2n-1}{2a}\pi x\right)\right)^2 dx}$$

$$\Rightarrow \quad C_n \sinh\left(\frac{2n-1}{2a}\pi b\right) = B_n$$

$$\Rightarrow C_n = \frac{B_n}{\sinh\left(\frac{2n-1}{2a}\pi b\right)}$$

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5. 2009.

$$\phi'(t) - \int_0^t (t-\tau)^2 \varphi(\tau) d\tau = \delta(t-3)$$

$$\varphi(0) = 1$$

$$\varphi'(t) - (t^2 * \varphi(t)) = \delta(t-3)$$

$$s\Phi(s) - \varphi(0) - \frac{2}{s^3} \cdot \Phi(s) = e^{-3s}$$

$$\Phi(s)\left(s - \frac{2}{s^3}\right) = e^{-3s} + 1$$

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