Lesson 5

01/21/2022

Last time: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$

E-values: $\lambda_1 = -3+4i$, $\lambda_2 = -3-4i$

Note: $\lambda_2 = \overline{\lambda_1}$ (if $\underline{\underline{A}}$ has real entries then any complex e-values come in conjugate pairs).

Find eigenvector assoc. to λ_1.

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{\underline{v}} = \underline{\underline{0}}$$

$$= \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -4iv_1 + 4v_2 = 0 & \Rightarrow \quad v_2 = iv_1 \quad (1) \\ -4v_1 - 4iv_2 = 0 & \Rightarrow \quad v_2 = -\frac{1}{i}v_1 \quad (2) \end{cases}$$

Notice: (1) & (2) give the same information:

$$\underbrace{v_2 = -\frac{1}{i}v_1}_{(2)} = -\frac{i}{i \cdot i}v_1 = iv_1$$

So: $v_2 = iv_1$

So an e-vector is: $\underline{\underline{v}}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

So a sol'n is $\underline{\underline{x}}_1 = e^{(-3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Need: a second lin. indep. sol'n.

1st way: Look at conjugate e-value λ_2 and find e-vector.

Before: $\underline{\underline{v}}_1$ e-vector assoc to λ_1 so:

$$\underline{\underline{A}}\,\underline{\underline{v}}_1 = \lambda_1 \underline{\underline{v}}_1$$

and $\lambda_2 = \bar{\lambda}_1$

Fact: $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ so:

$$\overline{\underline{\underline{A}}\,\underline{\underline{v}}_1} = \overline{\lambda_1 \underline{\underline{v}}_1} \qquad \text{(conjugates entry by entry)}$$

$$\Rightarrow \quad \bar{\underline{\underline{A}}}\,\bar{\underline{\underline{v}}}_1 = \bar{\lambda}_1 \underline{\underline{v}}_1 \qquad (\underline{\underline{A}} \text{ has real entries so } \underline{\underline{A}} = \bar{\underline{\underline{A}}})$$

$$\Rightarrow \quad \underline{\underline{A}}\,\bar{\underline{\underline{v}}}_1 = \lambda_2 \bar{\underline{\underline{v}}}_1$$

$\Rightarrow$ $\bar{\underline{\underline{v}}}_1$ is an eigenvector for λ_2

So: an eigenvector for $\lambda_2 = -3-4i$ is

$$\underline{\underline{v}}_2 = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

A second lin. indep. sol'n:

$$\underline{\underline{x}}_2(t) = e^{(-3-4i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Gen. sol'n:

$$\underline{\underline{x}}(t) = c_1 e^{(-3+4i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-3-4i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$c_1, c_2 \in \mathbb{C}$

2nd way to get a second sol'n:

Observation: $\underline{\underline{x}}$ solves:

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}, \quad \underline{\underline{A}} \text{ real entries}$$

Take real part (entry by entry)

$$\mathrm{Re}(\underline{\underline{x}}') = \mathrm{Re}(\underline{\underline{A}}\,\underline{\underline{x}})$$

Fact: if $c \in \mathbb{R}$ then $\mathrm{Re}(cz) = c\,\mathrm{Re}(z)$
and $\mathrm{Im}(cz) = c\,\mathrm{Im}(z)$

So:

$$\left(\mathrm{Re}(\underline{\underline{x}})\right)' = \underline{\underline{A}}\,\mathrm{Re}(\underline{\underline{x}})$$ bec. $\underline{\underline{A}}$ has real entries.

So if $\underline{\underline{x}}$ is a sol'n $\Rightarrow$ $\mathrm{Re}\,\underline{\underline{x}}$ is a sol'n

In the same way: $\mathrm{Im}\,\underline{\underline{x}}$ is also a sol'n.

So: if $\mathrm{Re}\,\underline{\underline{x}}$, $\mathrm{Im}\,\underline{\underline{x}}$ are linearly indep.
then general sol'n to $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is

given by

$$\underline{\underline{x}}(t) = a_1 \operatorname{Re} \underline{\underline{x}}(t) + a_2 \operatorname{Im} \underline{\underline{x}}(t),$$

where $\underline{\underline{x}}(t)$ solves $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ w/ $\underline{\underline{A}}$ having real entries.

Here $a_1, a_2 \in \mathbb{C}$, if we have real initial data then $a_1, a_2 \in \mathbb{R}$

Now: find Re, Im of $\underline{\underline{x}}(t) = e^{(-3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$

Use Euler's formula (Oh-ee-ler)

If $a, b \in \mathbb{R}$ then

$$e^{a+ib} = e^{a}\left(\cos(b) + i\sin(b)\right)$$

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Note:

$$e^{a-ib} = e^{a}\left(\cos(-b) + i\sin(-b)\right)$$

$$= e^{a}\left(\cos(b) - i\sin(b)\right)$$

Now: $\underline{\underline{x}}(t) = e^{(-3+4i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$$= \underbrace{e^{-3t+4ti}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-3t}\left(\cos(4t) + i\sin(4t)\right)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t}\cos(4t) + i\, e^{-3t}\sin(4t) \\ e^{-3t}\cos(4t)\, i - e^{-3t}\sin(4t) \end{bmatrix}$$

$\overset{\uparrow}{i^2}$

$$= \begin{bmatrix} e^{-3t}\cos(4t) \\ -e^{-3t}\sin(4t) \end{bmatrix} + i \begin{bmatrix} e^{-3t}\sin(4t) \\ e^{-3t}\cos(4t) \end{bmatrix}$$

<u>So</u>:

$$\mathrm{Re}(\underline{x}_1(t)) = \begin{bmatrix} e^{-3t}\cos(4t) \\ -e^{-3t}\sin(4t) \end{bmatrix}$$

$$\mathrm{Im}(\underline{x}_1(t)) = \begin{bmatrix} e^{-3t}\sin(4t) \\ e^{-3t}\cos(4t) \end{bmatrix}$$

Check linear indep.:

$$W\big(\mathrm{Re}(\underline{x}_1(t)), \mathrm{Im}(\underline{x}_1(t))\big) = \begin{vmatrix} e^{-3t}\cos(4t) & e^{-3t}\sin(4t) \\ -e^{-3t}\sin(4t) & e^{-3t}\cos(4t) \end{vmatrix}$$

$$= e^{-6t}\cos^2(4t) + e^{-6t}\sin^2(4t) = e^{-6t} \neq 0$$

$\Rightarrow$ lin. independent on $\mathbb{R}$.

Gen. sol'n:

$$\underline{\underline{x}}(t) = a_1 e^{-3t} \begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix} + a_2 e^{-3t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix}$$

or

$$x_1(t) = a_1 e^{-3t} \cos(4t) + a_2 e^{-3t} \sin(4t),$$
$$x_2(t) = -a_1 e^{-3t} \sin(4t) + a_2 e^{-3t} \cos(4t).$$

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Summary: $\underline{x}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}}$ cplx e-values, real entries.

→ Find eigenvalues.

→ Find an eigenvector $\underline{\underline{v}}_1$ cor. to one of them (λ_1)

→ A complex valued sol'n is $\underline{\underline{x}}(t) = e^{\lambda_1 t}\underline{\underline{v}}_1$.

→ Take real & imaginary pt of $\underline{\underline{x}}(t)$ using Euler's formula to obtain 2 real valued sol's.

Sol's of linear 2×2 systems from a geometric point of view.

$\underline{A} \to 2\times 2$, real entries, const.

Linear system:

$$\underline{x}' = \underline{A}\,\underline{x}. \quad (*)$$

Can think of a particular sol'n $\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ of $(*)$ as a curve in the plane.

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \leftrightarrow (x(t), y(t)), \quad t \in \mathbb{R}$$

Ex:

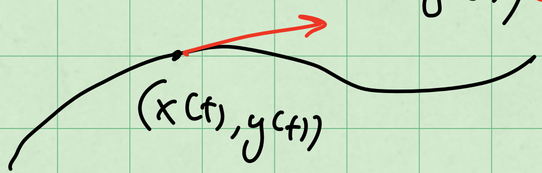
$$\underline{x}(t) = e^{-3t}\begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix}$$

is a sol'n to $\underline{x}' = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}\underline{x}'$,

can identify w/ curve

$$(x(t), y(t)) = \left(e^{-3t}\cos(4t), -e^{-3t}\sin(4t)\right).$$

$(x'(t), y'(t)) \leftarrow$ velocity vector.

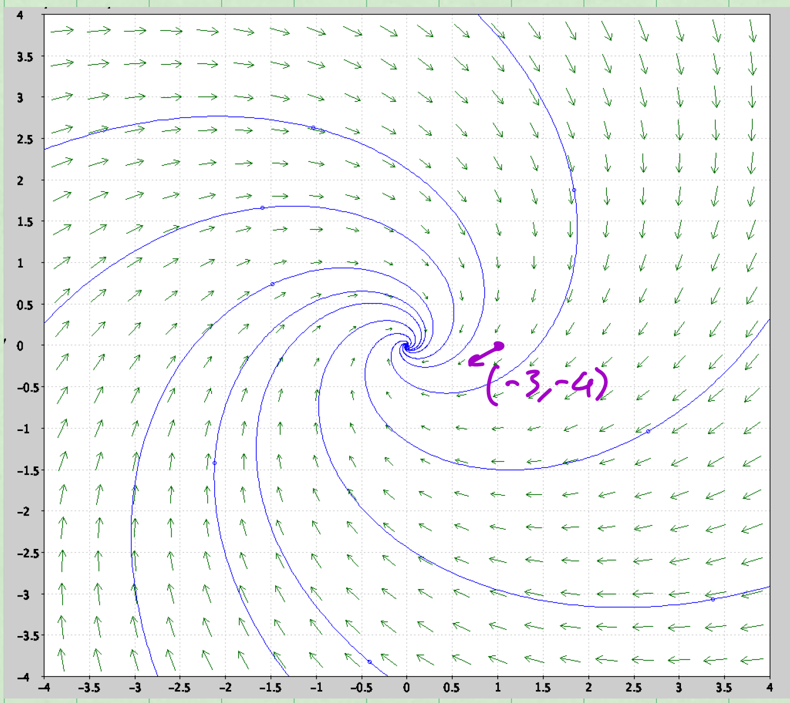


System $\underline{x}' = \underline{A}\,\underline{x}$ gives us the velocity vector of the cor. curve at each position.

Ex: velocity vector of curve passing through $(1, 0)$ for $\underline{x}' = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}\underline{x}$

is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

Finding velocity vectors at each $\underline{x}$ we can plot the phase plane portrait of system.



$$\underline{x}' = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} \underline{x}$$

Arrows: velocity vectors

Curves: solution curves.

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