Lesson 6

01/24/2022

§5.5 Linear systems with defective eigenvalues

Ex 1: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Want: 2 lin. indep. sols.

Find e-values:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2$$

$\Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$ multiplicity 2.

Seek eigenvectors

$$(\underline{\underline{A}} - 1\cdot \underline{\underline{I}})\underline{\underline{v}} = \underline{\underline{0}} \Rightarrow \underline{\underline{0}}\,\underline{\underline{v}} = \underline{\underline{0}}$$

Any non-zero vector is an e-vector.
In particular, we can find 2 lin indep. e-vectors.

Eg: $\underline{\underline{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{\underline{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ We can built 2 lin. indep. sols as

$$\underline{\underline{x}}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{\underline{x}}_2(t) = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ gen. sol'n:

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad //$$

Ex 2:

$$\underline{x}' = \underbrace{\begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}}_{\underline{A}} \underline{x}$$

Eigenvalues: $\lambda = 5$, multiplicity 2

Eigenvectors: $(\underline{A} - 5\underline{I})\underline{v} = \underline{0}$

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -4v_1 - 4v_2 = 0 \\ \cancel{4v_1 + 4v_2 = 0} \end{cases}$$

$$\Rightarrow \quad v_1 = -v_2$$

Can take

$$\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\Rightarrow$ a sol'n of $\underline{x}' = \underline{A}\,\underline{x}$ is

$$\underline{x}(t) = e^{5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Q: How do we find a second lin. indep. sol'n?

Method above doesn't give a second

lin. indep. sol'n.

A: Look for sol'n in a different form.

Terminology: If λ is an eigenvalue of $\underline{\underline{A}}$ w/ multiplicity k and we can find at most $p \le k$ lin. indep. eigenvectors associated w/ λ then we say that λ has defect $k-p$.

In Ex 1: defect $= 2-2 = 0$

Ex 2: defect $= 2-1 = 1$

If defect ≥ 1 then λ is called defective

To deal w/ eigenvalues of defect 1.

Assume that a sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ has the form

$$\underline{\underline{x}}(t) = e^{\lambda t}\left(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2\right)$$

w/ $\underline{\underline{v}}_1 \neq \underline{\underline{0}}$.

Plug in:

$e^{\lambda t} \neq 0$

$$\lambda \cancel{e^{\lambda t}}\left(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2\right) + \cancel{e^{\lambda t}}\,\underline{\underline{v}}_1 = \underline{\underline{A}}\left(\cancel{e^{\lambda t}}\left(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2\right)\right)$$

reorganize $\Rightarrow$

$$(\lambda \underline{v}_1 - \underline{A}\, \underline{v}_1)\, t + (\lambda \underline{v}_2 + \underline{v}_1 - \underline{A}\, \underline{v}_2) = 0$$

for all t.

$$\Rightarrow \begin{cases} (\underline{A} - \lambda \underline{I})\, \underline{v}_1 = \underline{0} & (1) \\ (\underline{A} - \lambda \underline{I})\, \underline{v}_2 = \underline{v}_1 \neq \underline{0} & (2) \end{cases}$$

(1) $\Rightarrow$ λ is an eigenvalue

$\underline{v}_1$ is an eigenvector

<u>Note:</u> $$(\underline{A} - \lambda \underline{I})^2 \underline{v}_2 = (\underline{A} - \lambda \underline{I})(\underline{A} - \lambda \underline{I})\underline{v}_2$$

$$= (\underline{A} - \lambda \underline{I})\, \underline{v}_1 = \underline{0}$$

So $\underline{v}_2$ satisfies

$$(*) \begin{cases} (\underline{A} - \lambda \underline{I})^2 \underline{v}_2 = \underline{0} \\ (\underline{A} - \lambda \underline{I})\, \underline{v}_2 \neq \underline{0} \end{cases}$$

A vector $\underline{v}_2$ which satisfies $(*)$ is called a generalized evector of rank 2.

<u>So:</u> to find a sol'n we should find

a generalized e-vector $\underline{v}_2$ by
(*) and an eigenvector $\underline{v}_1$ tied
tes $\underline{v}_2$ via (2).

Back to our Ex 2:

Look for gen. e-vector.

Want: $\underbrace{\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}}_{(\underline{A} - 5\underline{I})}^2 \underline{v}_2 = \underline{0}$ (1)

AND

$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \underline{v}_2 \neq \underline{0}$: (2)

Note: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So (1) $\Rightarrow$ gives no restrictions on $\underline{v}_2$: any restriction will come from (2)

Want: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \underline{v}_2 \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Can take $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 25 \\ -2 \end{bmatrix}, \ldots$

Can't take $\underline{v}_2$ to be an eigenvector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ won't work.

E.g. take $\underline{\underline{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$$\underline{\underline{v}}_1 = (A - 5I)\, \underline{\underline{v}}_2 \qquad (3)$$

$$= \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$\underline{\underline{v}}_1$ is an eigenvector but not any e-vector: it is tied to $\underline{\underline{v}}_2$ by means of (3)

A sol'n of $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ is given by

$$\underline{\underline{x}}_2(t) = e^{5t}\left(\begin{bmatrix} -8 \\ 8 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right),$$

and it is lin. indep. from $x_1 = e^{5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The general sol'n can be written as

$$\underline{\underline{x}}(t) = c_1\, \underline{\underline{x}}_1(t) + c_2\, \underline{\underline{x}}_2(t)$$

$$= c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t}\left(\begin{bmatrix} -8 \\ 8 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

Again: $\underline{x}(t) = e^{5t}\left(\begin{bmatrix}-1\\1\end{bmatrix}t + \begin{bmatrix}1\\1\end{bmatrix}\right)$ is

not a sol'n for $\underline{x}' = \underline{A}\,\underline{x}$

even though $\begin{bmatrix}1\\1\end{bmatrix}$ is a generalized

e-vector and $\begin{bmatrix}-1\\1\end{bmatrix}$ is an e-vector

bec. $\begin{bmatrix}-1\\1\end{bmatrix} \neq \begin{bmatrix}-4 & -4\\4 & 4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$.

Summary of method for systems w/ an e-value of defect 1.

1. Find eigenvalue(s) and corresponding e-vectors
2. For a defective eigenvalue λ, find a generalized e-vector of rank 2, i.e. solve

$$\begin{cases}(\underline{A} - \lambda \underline{I})^2 \underline{v}_2 = \underline{0} \\ (\underline{A} - \lambda \underline{I})\underline{v}_2 \neq \underline{0}\end{cases}$$

$\underline{v}_2 \neq \underline{0}$

Set $\underline{v}_1 = (\underline{A} - \lambda \underline{I})\underline{v}_2$

$x(t) = e^{\lambda t}(\underline{v}_1 t + \underline{v}_2)$ is a sol'n.

$e^{\lambda t}\underline{v}_1$ is also a sol'n, lin. independent.