Lesson 7

01/26/2022

Today: systems w/ eigenvalues of high defect

See also: handout in calendar

Recall: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ ($\underline{\underline{A}}$ 2x2), λ e-value of defect 1.

Found $\underline{\underline{v}}_2 \neq 0$ such that

$$\begin{cases} (\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2 = \underline{\underline{0}} \\ (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \neq 0 \end{cases}$$

eigenvector

Then:

$$x_1(t) = e^{\lambda t}\underline{\underline{v}}_1, \quad x_2(t) = e^{\lambda t}(\underline{\underline{v}}_1 t + \underline{\underline{v}}_2)$$

are 2 lin. independent sol's.

Idea: couldn't "fill up" the multiplicity of λ with enough true eigenvectors, so we used generalized ones.

Ex: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$,

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Find eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^4$$

So $(2-\lambda)^4 = 0 \Rightarrow \lambda = 2$ e-value w/ multiplicity 4.

Find eigenvectors:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{A} - 2\underline{I}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow b = c = 0$

E-vectors: $\begin{bmatrix} a \\ 0 \\ 0 \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

E.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are a pair of lin. indep. e-vectors.

or: $\underline{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ also lin. indep.

<u>Sol's:</u> $e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e^{2t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

<u>defect</u>: $4 - 2 = 2$

4: multiplicity; 2: # lin. indep. e-vectors

<u>Need</u>: "fill up" the multiplicity w/ 2 addition generalized e-vectors.

<u>Terminology</u>:
A generalized eigenvector of rank r for a matrix A assoc. to an eigenvalue λ is a non-zero vector $\underline{v}$ such that

$$\begin{cases} (\underline{A} - \lambda \underline{I})^{r} \underline{v} = 0 \\ (\underline{A} - \lambda \underline{I})^{r-1} \underline{v} \neq 0 \end{cases}$$

Eg: $r = 1 \rightarrow$ usual eigenvector.

<u>Def</u>: A length k chain of generalized e-vectors based on eigenvector v_1 is a set of vectors $\{\underline{v}_1, \dots, \underline{v}_k\}$ such that

$$(\underline{A} - \lambda \underline{I}) \underline{v}_k = \underline{v}_{k-1}$$
$$(\underline{A} - \lambda \underline{I}) \underline{v}_{k-1} = \underline{v}_{k-2}$$
$$\dots$$
$$(\underline{A} - \lambda \underline{I}) \underline{v}_2 = \underline{v}_1 \rightarrow \text{eigenvector.}$$

Note: v_k is an eigenvector of rank k:

e.g. for $\underline{\underline{v}}_3$:

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^3 \underline{\underline{v}}_3 = (\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_3$$

$$= (\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2$$

$$= (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_1 = \underline{\underline{0}} \text{ bec. } \underline{\underline{v}}_1 \text{ e-vector.}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_3 = \underline{\underline{v}}_1 \neq 0 \quad \text{bec. } \underline{\underline{v}}_1 \text{ e-vector}$$

To build sol's to $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$:

$$\underline{\underline{x}}_1 = e^{\lambda t} \underline{\underline{v}}_1$$

$$\underline{\underline{x}}_2 = e^{\lambda t} (\underline{\underline{v}}_1 t + \underline{\underline{v}}_2)$$

$$\underline{\underline{x}}_3 = e^{\lambda t} \left(\underline{\underline{v}}_1 \frac{t^2}{2!} + t \underline{\underline{v}}_2 + v_3\right)$$

$$\underline{\underline{x}}_k = e^{\lambda t} \left(\underline{\underline{v}}_1 \frac{t^{k-1}}{(k-1)!} + \dots + \underline{\underline{v}}_{k-1} t + \underline{\underline{v}}_k\right)$$

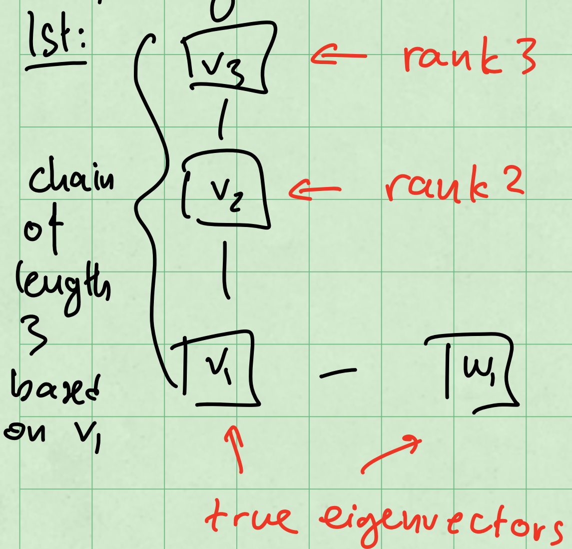
Back to example:

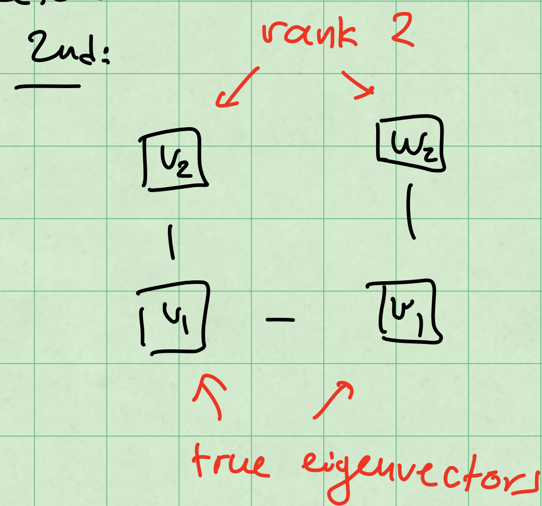
$\lambda = 2$: multiplicity 4, defect 2.

Need : total of 4 vectors, generalized & true eigenvector

2 → tre , 2 generalized.

2 possible configurations for chains of generalized e-vectors:





Rmk: v_1, w_1 are not necessarily the same eigenvectors we picked in the beginning but they are linear combinations of them.

Which conf. should we have? Unclear.

Seek chain of maximum possible length, defect + 1 = 3.

How: look for generalized e-vector of rank 3.

If we can find it, go down the chain to find $\underline{\underline{v}}_2$, $\underline{\underline{v}}_1$.

Do not start from true eigenvector and try to work up to find a chain. Do top $\rightarrow$ bottom.

<u>In the ex:</u> Want:

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^3 \underline{\underline{v}}_3 = \underline{\underline{0}}$$

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^2 \underline{\underline{v}}_3 \neq \underline{\underline{0}}.$$

<u>Compute:</u> $$(\underline{\underline{A}} - 2\underline{\underline{I}})^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\underline{\underline{A}} - 2\underline{\underline{I}})^3 = \underline{\underline{0}}$$

Want: $$\underline{\underline{0}}\, \underline{\underline{v}}_3 = \underline{\underline{0}} \quad (1)$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

(1) $\to$ no restriction

(2) $\Rightarrow$ $c \neq 0$

Can take e.g. $\underline{\underline{v}}_3 = \begin{bmatrix} 3 \\ 1 \\ 7 \\ 1 \end{bmatrix}$

Build rest of chain from $\underline{\underline{v}}_3$:

$$\underline{\underline{v}}_2 = (\underline{\underline{A}} - 2\underline{\underline{I}})\underline{\underline{v}}_3$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{v}}_1 = (\underline{\underline{A}} - 2\underline{\underline{I}})\underline{\underline{v}}_2 \overset{!}{=} (\underline{\underline{A}} - 2\underline{\underline{I}})^2 \underline{\underline{v}}_3$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

true eigenvector

Sols:

$$\underline{x}_1 = e^{2t}\begin{bmatrix}7\\0\\0\\0\end{bmatrix}$$

$$\underline{x}_2 = e^{2t}\left(\begin{bmatrix}7\\0\\0\\0\end{bmatrix}t + \begin{bmatrix}1\\7\\0\\0\end{bmatrix}\right)$$

$$\underline{x}_3 = e^{2t}\left(\begin{bmatrix}7\\0\\0\\0\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}1\\7\\0\\0\end{bmatrix}t + \begin{bmatrix}3\\1\\7\\1\end{bmatrix}\right)$$

A fourth sol'n: use an eigenvector lin. indep. from $\begin{bmatrix}7\\0\\0\\0\end{bmatrix}$ e.g.

$$\underline{x}_4 = e^{2t}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

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