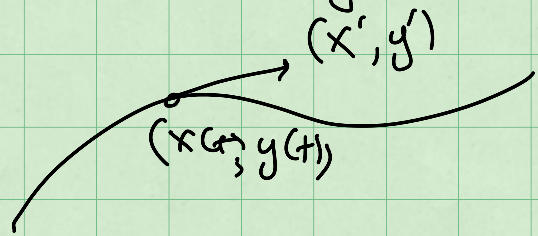
Lesson 8 $\qquad$ 01/28/2022

§ 5.3 Phase plane portraits (plot of velocity vectors of solution curves of 2×2 linear systems)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

Velocity vector at position $\begin{bmatrix} x \\ y \end{bmatrix}$ is determined by A & $\begin{bmatrix} x \\ y \end{bmatrix}$.

Today: Examine how the e-values of a linear 2×2 system affect the phase plane portrait.

1. 2 real e-values of opposite signs.

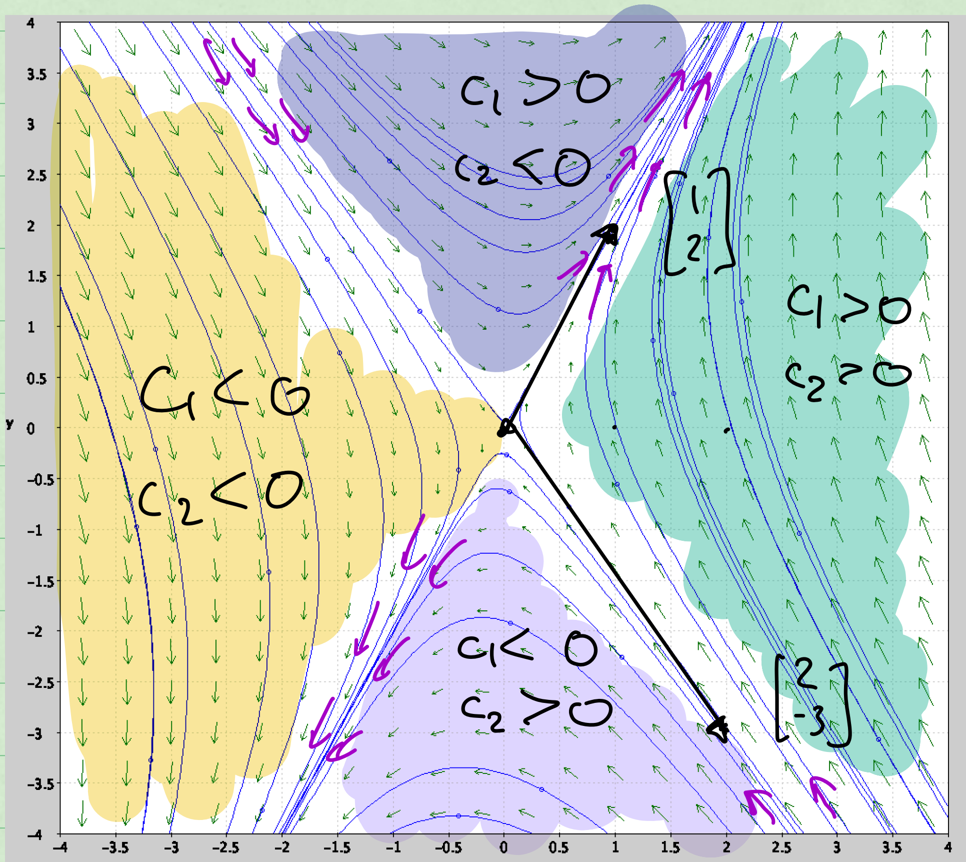
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5/7 & 6/7 \\ 18/7 & -2/7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues: $1, -2$

General sol'n:

eigenvectors

$$\underset{\sim}{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$



(made w/ pplane 8)

4 quadrants determined by eigenvectors. Sol's stay within quadrants

At time $t = 0$: $\quad \underline{\underline{x}}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

If $c_1, c_2 > 0$, $\underline{\underline{x}}(0)$ in green area and bec. $e^t, e^{-2t} > 0$, the entire sol'n curve will be in green area.

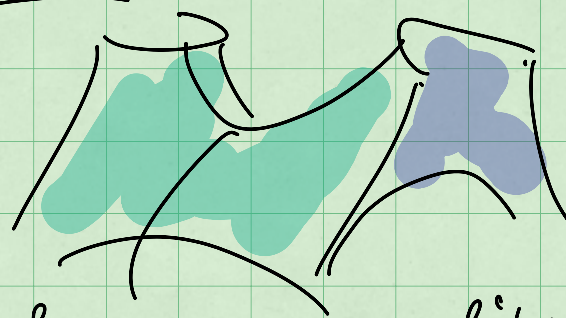
<u>Velocity at t:</u>

$$\underline{\underline{x}}'(t) = c_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 c_2 e^{-2t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

As $t \to \infty$, velocity tends to become $\parallel$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For a system w/ 2 real eigenvalues of opposite signs the origin is a <u>saddle point</u>.

$z = x^2 - y^2$:



level sets of z look like sol'n curves

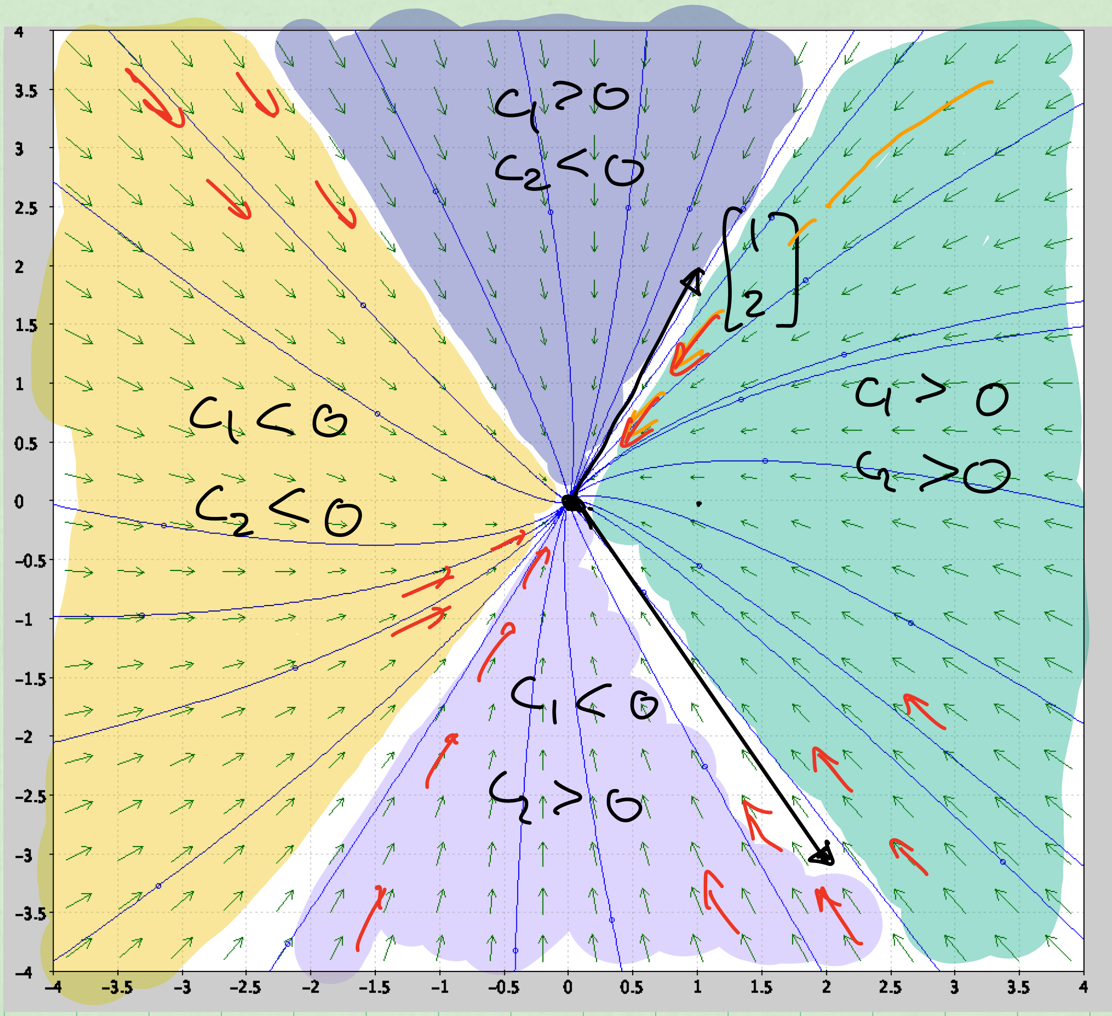
2. 2 distinct negative e-values.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{25}{7} & \frac{2}{7} \\ \frac{6}{7} & -\frac{27}{7} \end{bmatrix}}_{= A_2} \begin{bmatrix} x \\ y \end{bmatrix}$$

e-values $-3, -4$.

eigenvectors

<u>Sol'n:</u> $$\underset{\sim}{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$



Nodal Sink.

Sol's confined in quadrants

<u>velocity:</u>

$$\underset{=}{x}'(t) = -3e^{-3t}c_1\begin{bmatrix}1\\2\end{bmatrix} - 4c_2\,e^{-4t}\begin{bmatrix}2\\-3\end{bmatrix}$$

$$= e^{-3t}\left(-3c_1\begin{bmatrix}1\\2\end{bmatrix} - \underbrace{4c_2e^{-t}\begin{bmatrix}2\\-3\end{bmatrix}}_{\to 0 \text{ as } t\to\infty}\right)$$

so as $t \to \infty$, x' tends to become $\parallel$ to $\begin{bmatrix}1\\2\end{bmatrix}$

As $t \to -\infty$:

$$\underline{x}'(t) = e^{-4t}\left(\underbrace{-3e^{t} c_1 \begin{bmatrix}1\\2\end{bmatrix}}_{\to 0 \text{ as } t\to -\infty} - 4c_2 \begin{bmatrix}2\\-3\end{bmatrix}\right)$$

$\underline{x}'$ tends to become $\parallel$ to $\begin{bmatrix}2\\-3\end{bmatrix}$

3. Distinct positive eigenvalues

$$\underline{x}' = \underline{A}_1 \underline{x}, \quad \underline{A}_3 = \begin{bmatrix} \frac{25}{7} & -\frac{2}{7} \\ -\frac{6}{7} & \frac{27}{7} \end{bmatrix}$$

E-values: 3, 4.

Note: $\underline{A}_3 = -\underline{A}_2$

Principle of time reversal

A const.

Given a sol'n to a system $\underline{x}' = \underline{A}\,\underline{x}$ then the vector valued function

$$\underline{y}(t) = \underline{x}(-t)$$

solves

$$\underline{y}' = -A\,\underline{y}$$

Why:

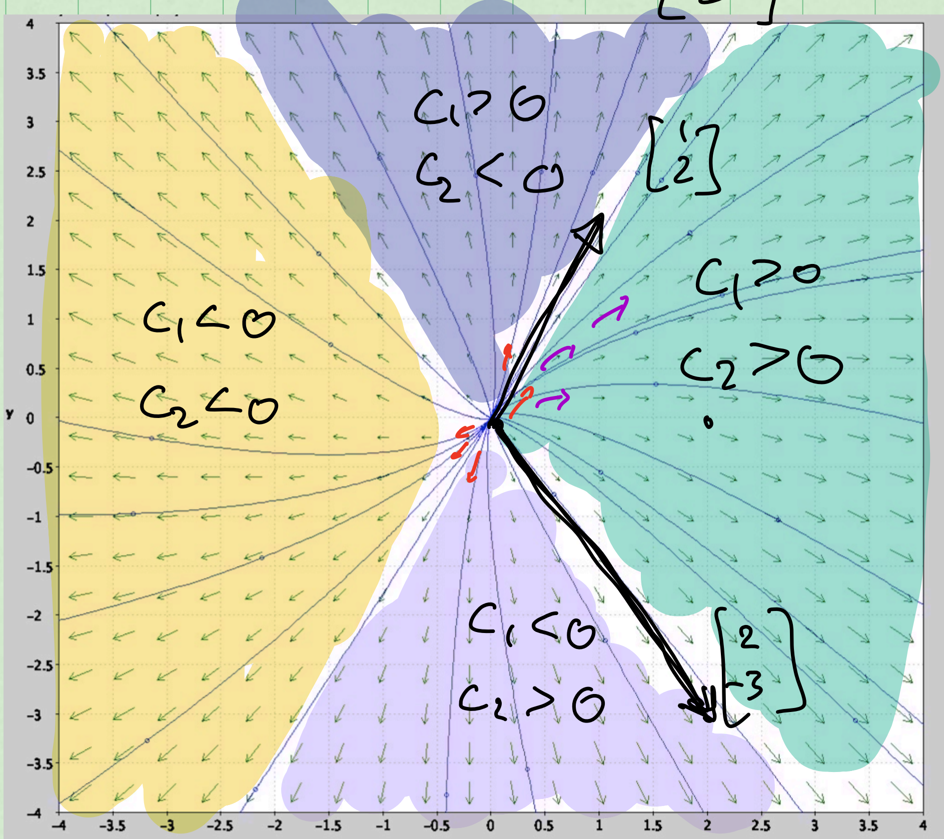
$$\frac{d}{dt}\left(\underline{y}(t)\right) = \frac{d}{dt}\left(\underline{x}(-t)\right)$$

$$= -\underline{x}'(-t) = -\underline{\underline{A}}\,\underline{x}(-t)$$

$$= -\underline{\underline{A}}\,\underline{y}.$$

So: In our example, the sol's to $\underline{x}' = \underline{\underline{A}}_3 \underline{x}$ are the same as those of $\underline{x}' = \underline{\underline{A}}_2 \underline{x}$ but with reversed time.

$$\underline{x}(t) = c_1 e^{+3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{+4t}\begin{bmatrix}2\\-3\end{bmatrix}$$



Phase plane portrait is same as before but w/ arrows pointing the other way.

Nodal source.

Terminology. The origin is a **node** if

1. either every trajectory approaches the origin as $t \to \infty$ [sink]
   or every trajectory recedes (goes away) from the origin as t increases [source]

AND

2. Every trajectory is tangent to a straight line through the origin at the origin.