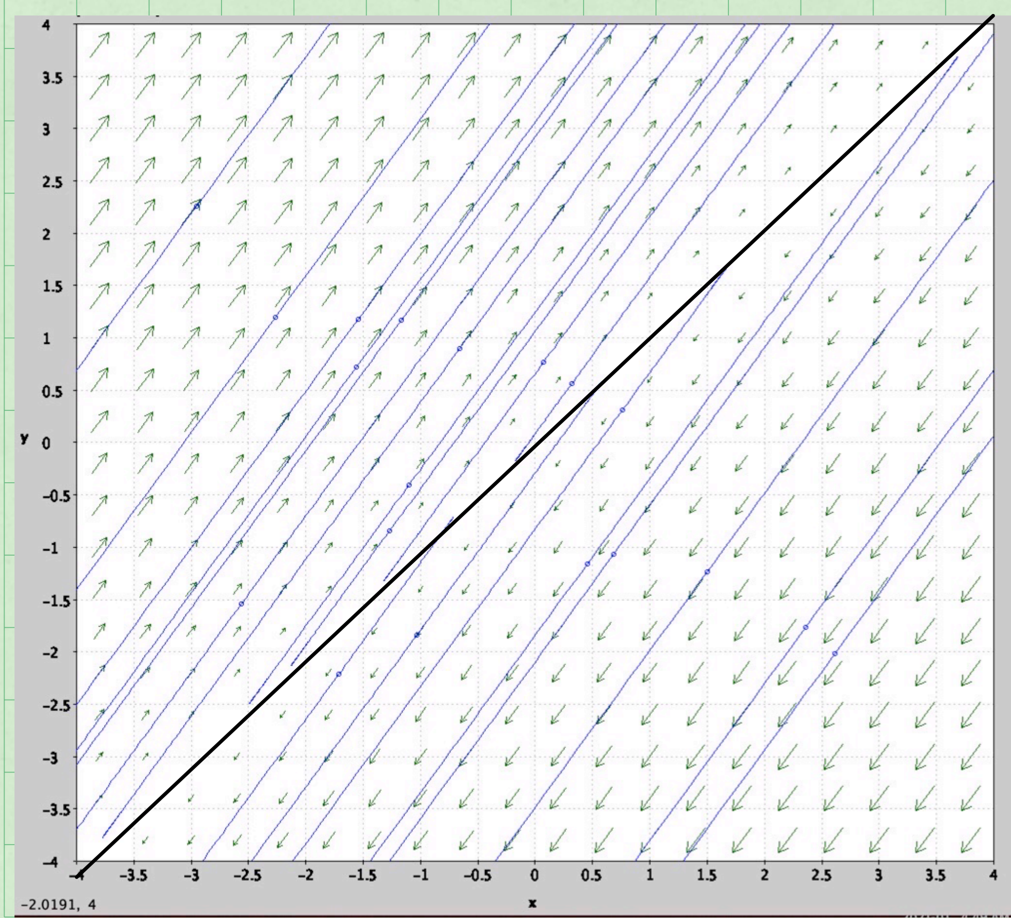
Lesson 9

01/31/2022

Were discussing phase plane portraits

Saw:

1. 2 real distinct e-values of opposite sign (saddle)
2. 2 distinct positive e-values (nodal source)
3. 2 distinct negative e-values (nodal sink)
4. 2 distinct real e-values, one 0: read textbook.



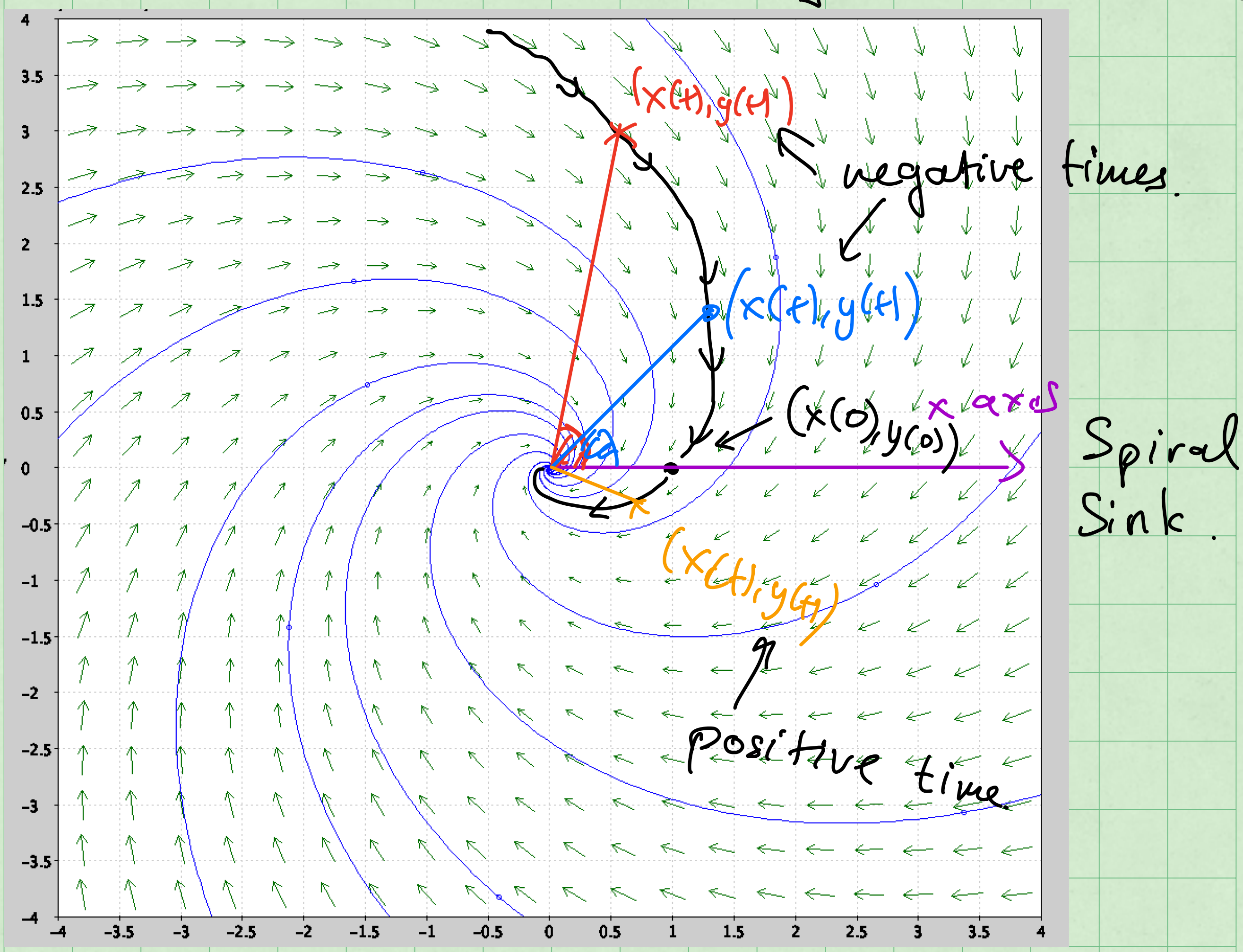
trajectories: half lines starting from a line which passes through origin

5. Complex eigenvalues w/ negative real part.

$$x' = -3x + 4y \qquad \lambda = -3 \pm 4i$$
$$y' = -4x - 3y$$

Sol'n:

$$\underline{x}(t) = a e^{-3t} \begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix} + b e^{-3t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \end{bmatrix}$$



Ex: Look at a specific trajectory:

e.g. the one w/ $\underline{x}(0) = (1,0)$

$a = 1$, $b = 0$

(in general: $\underline{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$)

$$\underline{x}(t) = e^{-3t} \begin{bmatrix} \cos(4t) \\ -\sin(4t) \end{bmatrix}$$

Distance of $\underline{x}(t)$ from the origin:

$$|\underline{x}(t)| = \sqrt{\left(e^{-3t}\cos(4t)\right)^2 + \left(-e^{-3t}\sin(4t)\right)^2}$$

$$= \sqrt{e^{-6t}\cos^2(4t) + e^{-6t}\sin^2(4t)}$$

$$= e^{-3t}$$

So: as $t \to \infty$, $\underline{x}(t)$ approaches origin exponentially fast.

Tangent of angle that is formed by line segment connecting origin with $(x(t), y(t))$ and positive x-axis is given by

$$\tan(\theta(t)) = \frac{y(t)}{x(t)}$$

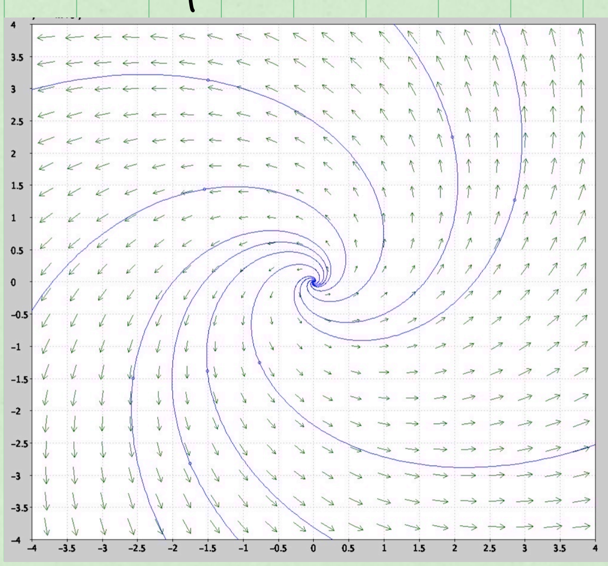
$$= \frac{-\cancel{e^{-3t}}\sin(4t)}{\cancel{e^{-3t}}\cos(4t)}$$

$$= -\tan(4t)$$

$\Rightarrow \quad \theta(t) = -4t \quad$ (up to addition of an integer multiple of π)

<u>So:</u> angle decreases as $t \to \infty$

<u>6</u> cplx e-values w/ positive real pt.:



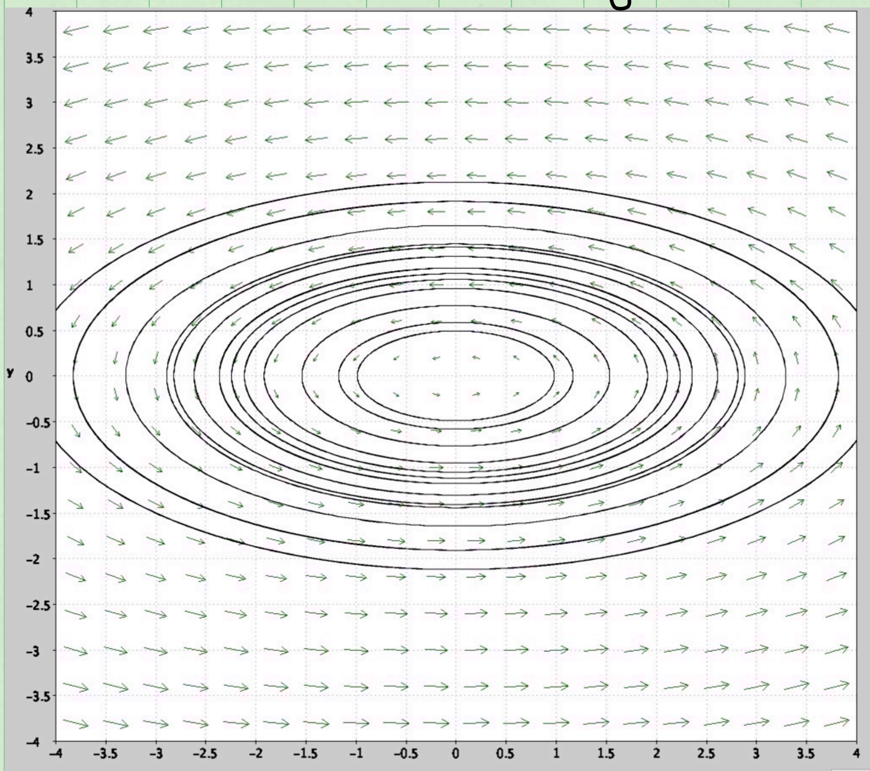
$$x' = 3x - 4y$$
$$y' = 4x + 3y$$

$\lambda = 3 \pm 4i$

By principle of time reversal, picture is qualitatively as in case 5 but w/ arrows reversed.

↑ spiral source.

7. Cplx eigenvalues w/ real pt 0
(purely imaginary)



trajectories are ellipses or circles

Origin called a center.

8 Repeated real eigenvalues

Ex: defect 0.

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

$$\frac{dx}{dt} = x \Rightarrow \int \frac{dx}{x} = \int dt$$
$$\ln x = t + c$$
$$x = e^c e^t = ae^t$$

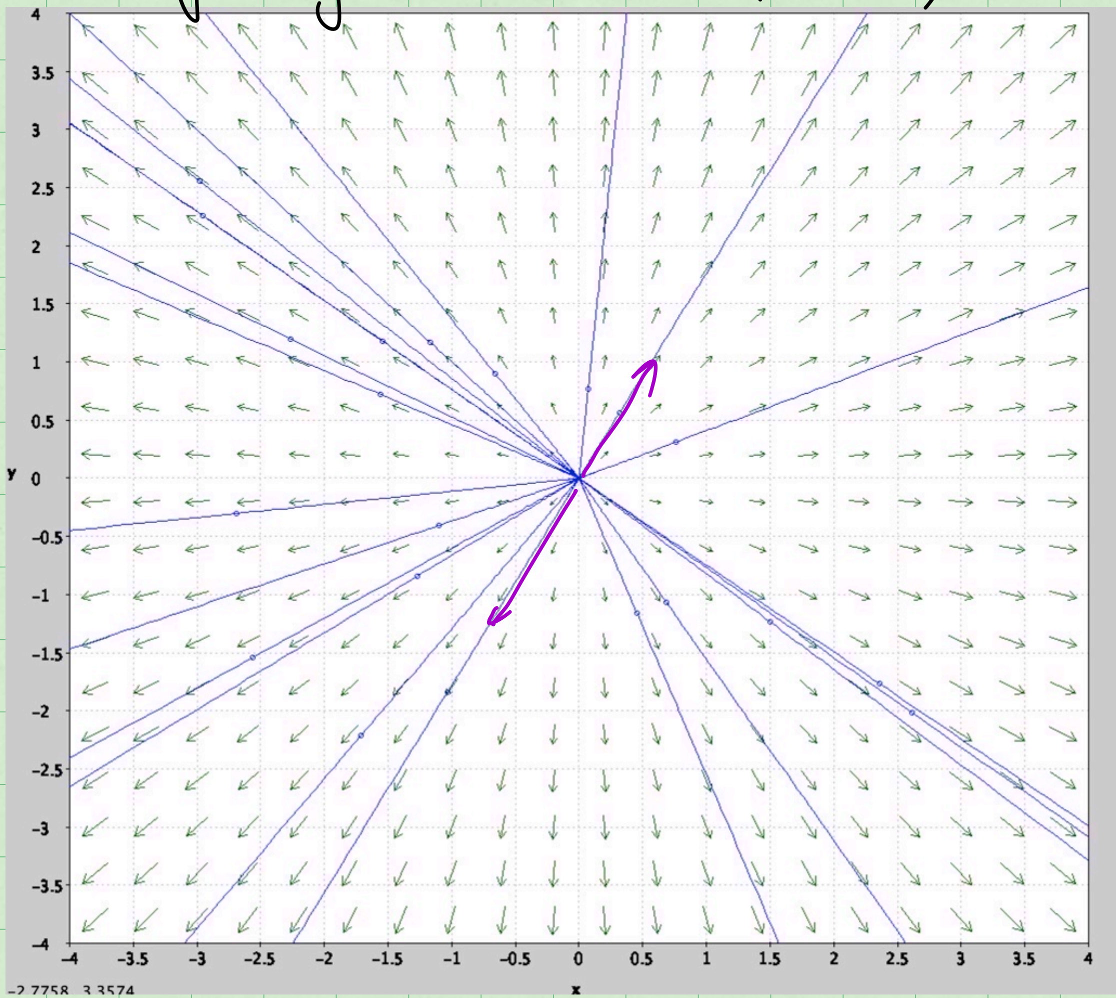
Gen. sol'n: $(x(t), y(t)) = (ae^t, be^t)$

Notice: if $a \neq 0$

$$\frac{y(t)}{x(t)} = \frac{b e^t}{a e^t} = \frac{b}{a}$$

$$\Rightarrow \underbrace{y(t) = \frac{b}{a} x(t)}$$

trajectory lies on the line $y = \frac{b}{a} x$ through the origin.
(trajectory is a half line)



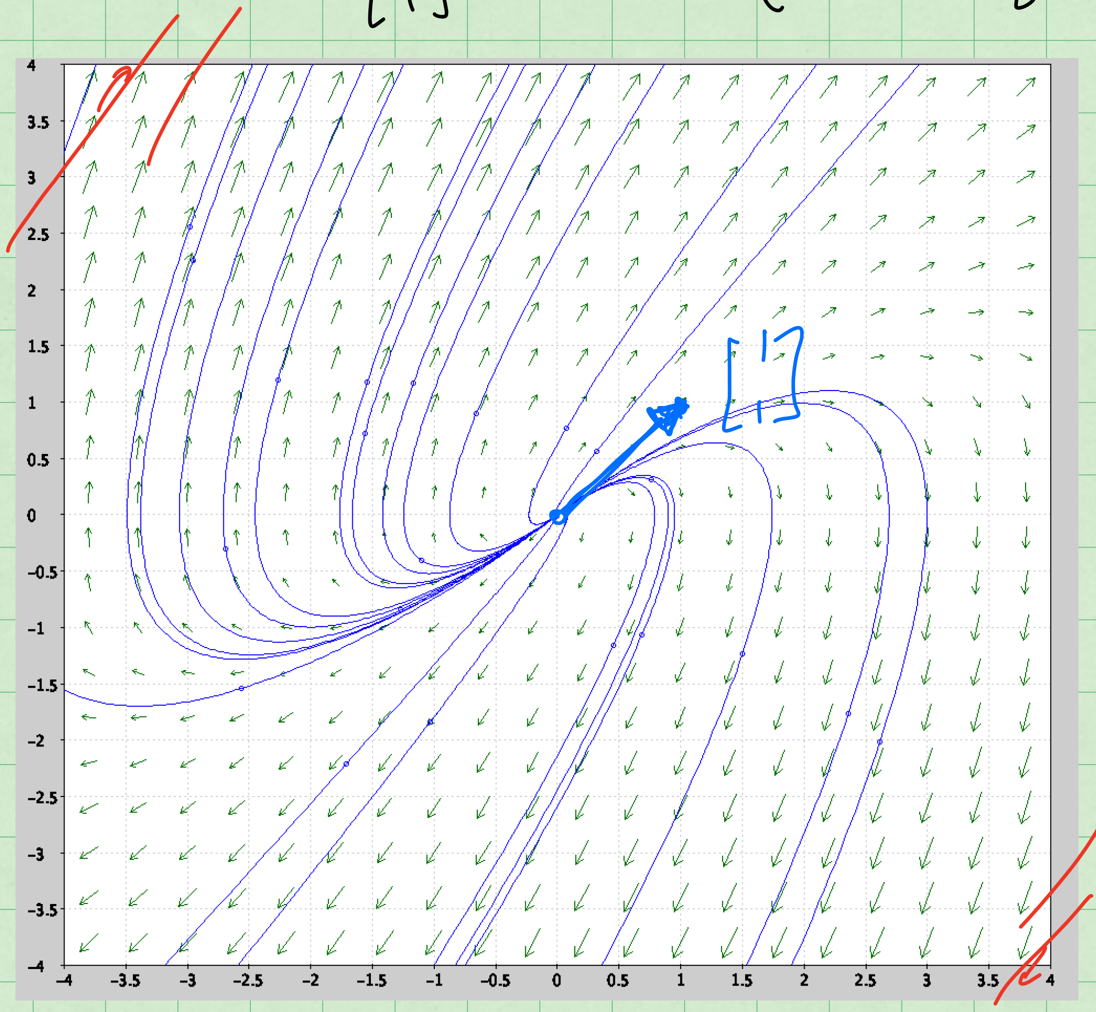
proper nodal source

9. Defect 1.

$$\begin{cases} x' = y \\ y' = -x + 2y \end{cases}$$

$\lambda = 1$ repeated e-value.

$$\underset{\sim}{x}(t) = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + b \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) e^{t}$$



Improper nodal source

Look at velocity:

$$\underline{x}'(t) = a\begin{bmatrix}1\\1\end{bmatrix}e^t + b\begin{bmatrix}1\\1\end{bmatrix}e^t + b\left(\begin{bmatrix}1\\1\end{bmatrix}t + \begin{bmatrix}3\\2\end{bmatrix}\right)e^t$$

What happens to velocity as $t \to \pm\infty$

<u>Write:</u>

$$\underline{x}'(t) = te^t\underbrace{\left(a\begin{bmatrix}1\\1\end{bmatrix}\cdot\frac{1}{t} + b\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{t} + b\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix}\frac{1}{t}\right)}$$

$$\longrightarrow b\begin{bmatrix}1\\1\end{bmatrix} \quad \text{as} \quad t \to \pm\infty$$

<u>So:</u> $\underline{x}'$ tends to become parallel to the true eigenvector $\begin{bmatrix}1\\1\end{bmatrix}$ as $t \to \pm\infty$
(direction determined by sign of b and whether $t \to \infty$ or $t \to -\infty$)

Node is called proper: if at most one pair of trajectories is tangent to the same line through the origin at the origin.
Improper otherwise.