Midterm 1 Practice Problems

Reminder: For the midterm you are allowed to use a double-sided, **handwritten**, $8.5 \times 11'$ (letter sized) note sheet.

Textbook Sections Covered:

- From Chapter 5: 5.1, 5.2, 5.3, 5.5
- From Chapter 6: 6.1, 6.2, 6.3, 6.4

Below, symbols in **boldface** indicate vectors or matrices. Primes indicate derivative with respect to t. The problems below offer an overview of the most important types of problems appearing in the sections above; *you are not expected to be able to complete all of the problems below in one hour.*

1. You are given the system

$$\begin{cases} x_1' = x_1 - 5x_2 \\ x_2' = x_1 + 3x_2 \end{cases}. \tag{1}$$

   (a) Find a pair of linearly independent solutions for (1). Each of the two solutions you find should be a pair of the form $(x_1(t), x_2(t))$ with $x_1(t)$ and $x_2(t)$ **real valued functions.** Make sure to check linear independendence as part of your solution.

   (b) Which of the phase plane portraits below corresponds to (1)?

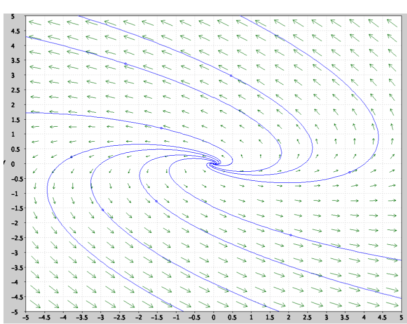


Figure 1: A.

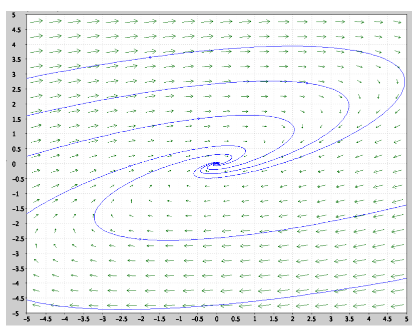


Figure 2: B.

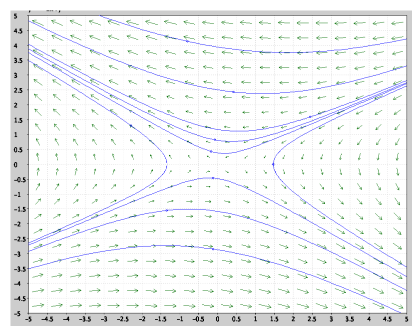


Figure 3: C.

2. Find the solution of the system

$$\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = 6x_1 - x_2 \end{cases}$$

   which satisfies $x_1(0) = 2$ and $x_2(0) = -1$.

3. You are given the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix}.$$

   Compute the general solution of the system. The following information is given:

- $\mathbf{A}$ has the eigenvalue $\lambda = 2$ with multiplicity 3.
- The following matrix powers are given:

$$\begin{bmatrix} -17 & -7 & 4 \\ 34 & 14 & -11 \\ 17 & 7 & 3 \end{bmatrix}^2 = \begin{bmatrix} 119 & 49 & 21 \\ -289 & -119 & -51 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -17 & -7 & 4 \\ 34 & 14 & -11 \\ 17 & 7 & 3 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. You are given the nonlinear system

$$\begin{cases} \frac{dx}{dt} = F(x,y) = ye^{x+y} \\ \frac{dy}{dt} = G(x,y) = -xe^{x+y} \end{cases} \tag{2}$$

   (a) Show that $(0,0)$ is a critical point for the system (2).

   (b) Compute the linearization of (2) at the origin. What does the phase plane portrait of the *linearized* system look like?

   (c) What does Part 4b imply about the phase plane portait of the *nonlinear* system (2) near the origin?

   (d) Solve the equation

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$$

   to find the trajectories of (2) in implicit form. Based on your answer, determine what the phase plane portrait of (2) looks like near the origin.

5. You are given the *linear* system

$$\begin{cases} \frac{dx}{dt} = 4x + \varepsilon y \\ \frac{dy}{dt} = x - y \end{cases} \tag{3}$$

   depending on a linear parameter ε. Determine the values of ε for which the phase plane portrait of (4) is an unstable node.

6. You are given the following system

$$\begin{cases} \frac{dx}{dt} = 7x - x^2 + xy \\ \frac{dy}{dt} = y - 4xy \end{cases} \tag{4}$$

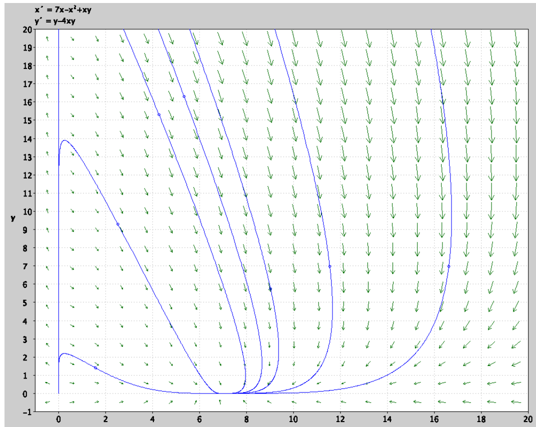
   describing two animal populations.

   (a) Is the relationship between them one of cooparation, predation or competition? In case of predation determine which population is the prey and which one is the predator.

   (b) Find the physically relevant critical points of the system.

   (c) Suppose we start with initial conditions $x = 2$ and $y = 4$. Based on the phase plane portrait below, what is the long term behavior of the system?

7. The equation $x'' + 4x - 5x^3 + x^5 = 0$ corresponds to a mass and spring system in which we retain the 3rd and 5th degree terms of the spring force function. Turn this equation into an equivalent 1st order system and determine all equilibrium solutions.



8. * It was mentioned in lecture that when you are trying to find a chain of generalized eigenvectors of length k based on a true eigenvector for a matrix A with eigenvalue λ of high defect you should start from the top instead of the bottom. That is, you should first find a generalized eigenvector $\mathbf{v}_k$ of the rank k and go down the "ladder"

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_k &= \mathbf{v}_{k-1} \\
&\dots \\
(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 &= \mathbf{v}_2 \\
(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1
\end{aligned} \tag{5}$$

until you reach a true eigenvector, instead of starting with a true eigenvector and trying to find a chain based on it by working your way up. The reason the "bottom-to-top" approach does not always work has to do with the fact that the matrix $\mathbf{A}-\lambda\mathbf{I}$ is singular, and the some of the equations $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_m = \mathbf{v}_{m-1}$ in (5) might not be solvable for $\mathbf{v}_m$. The following exercise demonstrates what can go wrong.

Suppose that we have the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(a) Check that $\lambda = 3$ is an eigenvalue of multiplicity 4 and defect 2.

(b) Check that $\mathbf{v}_1 = [0,0,0,1]^T$ is an eigenvector.

(c) What happens if you try to find a chain of generalized eigenvectors based on $\mathbf{v}_1$ by starting from $\mathbf{v}_1$ and trying to work your way up using (5)?

(d) Find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. You are given the following:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$