MA 30300
Midterm 2 Review Worksheet

- Sections covered: 7.1-7.6, 9.1-9.2.
  In this worksheet Problems 13-16 are from 9.3 and you do not need to worry about them for this exam.
- The Laplace transform table as it appears in p. 781 of the textbook will be provided.

1. Compute the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$, where $f(t) = \sinh(t)$ from the definition, i.e. without using a table (recall that $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$). For what s is $F(s)$ defined?

2. Use the Laplace Transform to solve the initial value problem
$$\begin{cases} x' = 2x + y \\ y' = 6x + 3y \end{cases}$$
under the initial conditions $x(0) = -1$, $y(0) = 2$.

3. Find the solution to the following initial value problem
$$\begin{cases} x''' + 4x'' + 4x' = e^{-2t} \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0. \end{cases}$$

4. You are given the following two functions defined for $t \geq 0$:
$$f(t) = \begin{cases} 1, & 0 \leq t \leq \pi \\ 0, & \text{otherwise} \end{cases} \quad , \quad g(t) = \cos(t).$$
Sketch their graphs and compute their convolution $f * g(t)$ for $t \geq 0$. Sketch the graph of the convolution as well.

5. You are given the following functions defined for $t \geq 0$
$$f_\alpha(t) = \cos(\alpha t), \quad g(t) = \cos(t),$$
where $\alpha \geq 0$ is a parameter.
   (a) Compute the convolution $f_\alpha * g(t)$ for $t \geq 0$, for all values of the parameter $\alpha \geq 0$.
   (b) For what values of α is $f_\alpha * g(t)$ bounded as a function of t?
   (c) For what values of α is $f_\alpha * g(t)$ a periodic function of t?
   Hint: When a function is periodic, any integer multiple of a period is also a period. If $\beta > 0$, what are the periods of the function $\sin(\beta t)$?

6. Compute the Laplace transform of the following functions, defined for $t \geq 0$:
   (a) $f(t) = \dfrac{e^t - e^{-t}}{t}$

(b) $g(t) = t^2 \cos(2t)$

(c) $h(t) = t^3$ if $1 \le t \le 2$, $h(t) = 0$ otherwise.

7. Compute the inverse Laplace transform of $F(s) = \arctan\left(\frac{3}{s+2}\right)$.
   Hint: First find $\mathcal{L}^{-1}\{F'(s)\}$.

8. Use properties of the Laplace Transform and the formulas

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt), \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$$

   to compute $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$.
   Note: The formula for $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$ is actually included in the Laplace transform table, but this exercise is asking you to derive it.

9. Solve the integrodifferential equation describing the current $i(t)$ in an RLC circuit given an impressed voltage $e(t)$:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = e(t), \quad i(0) = 0,$$

   where

$$L = 1, \quad R = 150, \quad C = 2 \times 10^{-4}, \quad e(t) = \begin{cases} 100t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}.$$

10. Solve the initial value problem

$$x'' + 2x' + x = \delta(t) - \delta(t-2), \quad x(0) = x'(0) = 2.$$

11. Find the weight function (unit impulse response) for the spring-mass system

$$mx'' + cx' + kx = f(t), \quad x(0) = x'(0) = 0,$$

    where $m = 1$, $c = 6$ and $k = 9$, and apply Duhamel's principle to write an integral formula for the solution in terms of the input f.

12. Which of the following functions are periodic on $\mathbb{R}$?

    (a) $f_1(t) = \tan(t)$ (assume it is defined to be 0 for the values of t where $\tan(t)$ is undefined)
    (b) $f_2(t) = \sinh(2t)$
    (c) $f_3(t) = t\sin(2t)$
    (d) $f_4(t) = \arctan(t+1)$
    (e) $f_5(t) = \sin(\pi t) + \sin(t)$

13. For each of the functions below defiened on $\mathbb{R}$, decide whether it is odd, even, or neither even nor odd.

(a) $f_1(t) = t^2 \cos(t)$

(b) $f_2(t) = t\cos(t)$

(c) $f_3(t) = (t+1)\sin(t)$

(d) $f_4(t) = t^2$

(e) $f_5(t) = \begin{cases} t^2, & t \geq 0 \\ -t^2, & t < 0 \end{cases}$

14. Compute the Fourier series for the following periodic functions (assume that their value at points of discontinuity is defined to be the average of their side limits there):

    (a) $f_1(t) = \begin{cases} 0, & -2 < t < 0 \\ t^2, & 0 < t < 2 \end{cases}$, periodic with period 4.

    (b) $f_2(t) = t^2, 0 < t < 2$, periodic with period 2.

    (c) $f_3(t) = \begin{cases} 0, & -1 < t < 0 \\ \sin(\pi t), & 0 < t < 1 \end{cases}$, periodic with period 2.

    Which of the functions above, if any, are even? Which ones are odd? For which ones is the term-by-term differentiation of the Fourier series valid?

15. For the following functions defined on intervals of the form $I = [0, L]$, sketch the graphs of their even and odd $2L-$periodic extensions. Then compute their Fourier sine and cosine series of the original functions (equivalently, the usual Fourier series of the even and odd periodic extensions, respectively):

    (a) $f_1(t) = \cos(t)$ on $I = [0, \pi]$

    (b) $f_2(t) = \cos(t)$ on $I = [0, 3\pi/2]$

16. Find a formal solution for the endpoint problem $x'' - 4x = 1, \quad x(0) = x(\pi) = 0$