Answers / Outlines of solutions

1. Do it for general a:

$$\int_0^\infty e^{-st} e^{at} dt = \frac{1}{s-a}, \quad s > a \qquad \text{(done in class)}$$

So $\mathcal{L}\{\sinh(t)\} = \frac{1}{2}\mathcal{L}\{e^{t}\} - \frac{1}{2}\mathcal{L}\{e^{-t}\} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$

$$= \frac{1}{s^2-1}$$

for $s > 1$

2. Taking Laplace:

$$sX(s) + 1 = 2X(s) + Y(s)$$
$$sY(s) - 2 = 6X(s) + 3Y(s)$$

$$\Rightarrow \begin{cases} (s-2)X(s) - Y(s) = -1 \\ -6X(s) + (s-3)Y(s) = 2 \end{cases}$$

$$\Rightarrow (s-2)(s-3)X(s) - (s-3)Y(s) = -(s-3)$$
$$-6X(s) + (s-3)Y(s) = 2 \qquad (+)$$

$$(s^2-5s)X(s) = -(s-3) + 2$$

$$\Rightarrow X(s) = \frac{-s+5}{s(s-5)} = -\frac{1}{s} \Rightarrow x(t) = -1$$

$$\Rightarrow y(t) = 2.$$

3. Use Laplace transform : $X(s) = \mathcal{L}\{x(t)\}$

$$s^3 X(s) + 4s^2 X(s) + 4s X(s) = \frac{1}{s+2}$$

$$\Rightarrow X(s) = \frac{1}{(s+2)^3 s}$$

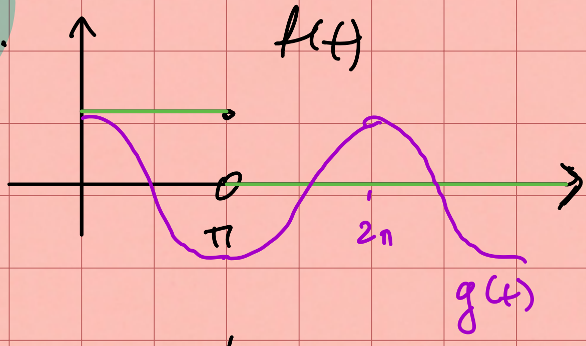
partial fractions:

$$\frac{s+3}{(s+2)^4 s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$\Rightarrow$ - - - - $\xRightarrow{\text{tables}}$

$$x(t) = \frac{1}{8} e^{-2t}\left(-1 - 2t - 2t^2\right) + \frac{1}{8}$$

4.



$$f * g(t) = \int_0^t f(\tau)\cos(t-\tau)\,d\tau = \begin{cases} \int_0^t \cos(t-\tau)\,d\tau & t \le \pi \\ \int_0^\pi \cos(t-\tau)\,d\tau & t > \pi \end{cases}$$

$$= \begin{cases} \sin(t) & t \leq \pi \\ 2\sin(t) & t > \pi \end{cases}$$

Or: $f * g = \mathcal{L}^{-1}\{\mathcal{L}\{f * g\}\}$

$= \mathcal{L}^{-1}\{\mathcal{L}\{f\}\mathcal{L}\{g\}\}$

$= \ldots$

5.a) $f_\alpha * g = \int_0^t \cos(\alpha\tau)\cos(t-\tau)\, d\tau$

$= \int_0^t \cos(\alpha\tau)\left(\cos(t)\cos(\tau) - \sin(t)\sin(\tau)\right) d\tau$

$= \cos(t)\int_0^t \cos(\alpha\tau)\cos(\tau)\, d\tau$

$- \sin(t)\int_0^t \cos(\alpha\tau)\sin(\tau)\, d\tau$

If $\alpha \neq 0$

$$\int_0^t \cos(\alpha\tau)\cos(\tau)\,d\tau = \frac{1}{\alpha}\sin(\alpha\tau)\cos(\tau)\Big|_0^t + \frac{1}{\alpha}\int_0^t \sin(\alpha\tau)\sin(\tau)\,d\tau$$

$$= \frac{1}{\alpha}\sin(\alpha t)\cos(t) - \frac{1}{\alpha^2}\cos(\alpha\tau)\sin(\tau)\Big|_0^t + \frac{1}{\alpha^2}\int_0^t \cos(\alpha\tau)\cos(\tau)\,d\tau$$

$$\Rightarrow \int_0^t \cos(\alpha\tau)\cos(\tau)\,dt \left(1-\frac{1}{\alpha^2}\right) = \frac{1}{\alpha}\sin(\alpha t)\cos(t) - \frac{1}{\alpha^2}\cos(\alpha t)\sin(t)$$

$\alpha \neq 1$

$$\Rightarrow \int_0^t \cos(\alpha\tau)\cos(\tau)\,d\tau =$$

$$\frac{\alpha}{\alpha^2-1}\sin(\alpha t)\cos(t) - \frac{1}{\alpha^2-1}\cos(\alpha t)\sin(t)$$

Similarly for the other integral. So

If $\alpha \neq 0, 1$

$f_\alpha * g =$

$$\cos(t)\left[\frac{\alpha}{\alpha^2-1}\sin(\alpha t)\cos(t) - \frac{1}{\alpha^2-1}\cos(\alpha t)\sin(t)\right]$$

$$-\sin^2(t)\,\frac{\sin(\alpha t)}{\alpha^2-1} - \sin(t)\cos(t)\,\frac{\cos(\alpha t)}{\alpha^2-1} - \frac{1}{\alpha^2-1}$$

If $\alpha = 0$:

$$f_\alpha * g = \sin(t)$$

If $\alpha = 1$:

$$f_\alpha * g = \frac{1}{2}\left(t\cos(t) + \sin(t)\right)$$

b). $f_\alpha * g$ bounded for all $\alpha \neq 1$

c) Not periodic for $\alpha = 1$
Periods of $\sin(\alpha t)$: $\frac{2\pi}{\alpha} n,\ n \in \mathbb{Z}$.

Periods of $\sin(t)$: $2\pi m,\ m \in \mathbb{Z}$.

In order to have $f_\alpha * g$ periodic we need

$$\frac{2\pi}{\alpha} n = 2\pi m \quad \text{for some } m, n \in \mathbb{Z}$$

$$\Rightarrow \alpha = \frac{n}{m} \quad \text{a rational number.}$$

6. a) $\log\left(\frac{1+s}{s-1}\right)$ (use Thm 3, p. 471)

b) $\dfrac{2s(s^2-12)}{(s^2+4)^3}$ (use Thm 2, p. 469)

c) Write as $h(t) = t^3\left(u(t-1) - u(t-2)\right)$

$$= \left((t-1)+1\right)^3 u(t-1) + \left((t-2)+2\right)^3 u(t-2)$$

$$H(s) = \frac{1}{s^4}\left[(s^3+3s^2+6s+6)e^{-s} - (8s^3+12s^2+12s+6)e^{-2s}\right]$$

7. Use rule $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$ (thm 2, p. 469)

$$f(t) = \frac{e^{-2t}\sin(3t)}{t}$$

8. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

Note: $\dfrac{1}{(s^2+k^2)} = -\dfrac{1}{2s}\dfrac{d}{ds}\left(\dfrac{1}{s^2+k^2}\right)$

So since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2s}\frac{d}{ds}\left(\frac{1}{s^2+k^2}\right)\right\}$$

$$= -\frac{1}{2}\int_0^t \mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+k^2}\right)\right\}d\tau$$

$$= -\frac{1}{2}\int_0^t (-\tau)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}d\tau$$

$$= \frac{1}{2k}\int_0^t \tau \sin(k\tau)\,d\tau$$

$$= -\frac{1}{2k^2}\int_0^t \tau\left(\cos(k\tau)\right)'d\tau$$

$$= -\frac{1}{2k^2}\left(t\cos(kt) - \int_0^t \cos(k\tau)\,d\tau\right)$$

$$= -\frac{1}{2k^2}\left(t\cos(kt) - \frac{1}{k}\sin(kt)\right)$$

9. Take Laplace:

$$\mathcal{L}\{e(t)\} = 100t\left(1-u(t-1)\right)$$

So:

$$sI(s) + 150\,I(s) + 5000\,\frac{I(s)}{s} = \frac{100}{s^2} - 100e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

$\Rightarrow$ ... $\Rightarrow$

$$I(s) = \frac{100}{s(s+50)(s+100)} + e^{-s}\frac{100(s+1)}{s(s+50)(s+100)}$$

Use p-fractions

$$i(t) = \frac{1}{50}\left[1-2e^{-50t} + e^{-100t}\right]$$

$$- \frac{1}{50}u(t-1)\left[1+98e^{-50(t-1)} - 99e^{-100(t-1)}\right]$$

10 Laplace:

$$\left(s^2X(s) - 2s - 2\right) + 2\left(sX(s) - 2\right) + X(s) = 1 - e^{-2s}$$

$$\left(s^2 + 2s + 1\right)X(s) = 7 + 2s - e^{-2s}$$

$$\Rightarrow X(s) = \frac{7+2s-e^{-2s}}{s^2+2s+1}$$

$$\Rightarrow X(s) = \frac{7+2s}{(s+1)^2} - e^{-2s}\frac{1}{(s+1)^2}$$

Write $\frac{7+2s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$

Find $A = 2$, $B = 5$

$$\Rightarrow X(s) = \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

$\Rightarrow \quad x(t) = (2+5t)e^{-t} - u(t-2)(t-2)e^{-(t-2)}$

11. $x'' + 6x' + 9x = f(t)$

$$\Rightarrow X(s)(s^2+6s+9) = F(s)$$

$$\Rightarrow X(s) = \frac{1}{s^2+6s+9} F(s)$$

$$\Rightarrow x(t) = \int_0^t w(\tau) f(t-\tau)\,d\tau \quad , \text{where}$$

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+9}\right\} = e^{-3t}t$$ is the weight function.

So $x(t) = \int_0^t e^{-3\tau}\tau f(t-\tau)\,d\tau$

12. Only a) is periodic:

$\sinh(2t) = \frac{1}{2}(e^{2t} - e^{-2t})$ is not periodic

$t\sin(2t)$ is cont. and increases unboundedly

$\arctan(t+1)$ is strictly increasing for all t.

$\sin(\pi t) + \sin(t)$:

↳ periods $2\pi, 4\pi, 6\pi$

↳ periods : $2, 4, 6, \ldots$

} no common period.

13. a) Even: $f_1(-t) = (-t)^2\cos(-t) = t^2\cos t = f_1(t)$

(or even × even = even)

b) Odd: $f_2(-t) = -t\cos(-t) = -t\cos(t)$

$= -f_2(t)$

(or odd × even = odd)

c) Neither $f_3(-t) = (1-t)\sin(-t)$

$= (-1+t)\sin(t)$

$\neq f_3(t)$ or $-f_3(t)$

d) Even $f_4(-t) = (-t)^2 = t^2 = f_4(t)$

e) Odd $f_5(-t) = \begin{cases} (-t)^2 & (-t) \geq 0 \\ -(-t)^2 & -t < 0 \end{cases}$

$= \begin{cases} t^2 & t \leq 0 \\ -t^2 & t > 0 \end{cases}$

$= -f_5(t)$

14. a) $a_0 = \frac{1}{2}\int_0^2 t^2\,dt = \frac{4}{3}$

$$a_n = \frac{1}{2}\int_0^2 t^2 \cos\frac{n\pi t}{2}\,dt = \frac{8(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\int_0^2 t^2 \sin\frac{n\pi t}{2}\,dt$$

$$= -\frac{4\left((n^2\pi^2-2)\cos(n\pi) - 2n\pi\sin(n\pi)+2\right.}{n^3\pi^3}$$

$$= \begin{cases} \frac{-4}{\pi n} & n \text{ even} \\ \frac{4}{n\pi} - \frac{16}{n^3\pi^3} & n \text{ odd.} \end{cases}$$

Plug these into series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{2}t\right) + b_n\sin\left(\frac{n\pi}{2}t\right)\right)$$

b) See textbook, ex. 2 p. 577.

c) $a_0 = \frac{2}{\pi}$, $a_n = -\frac{1+(-1)^n}{\pi(n^2-1)}$, $n>1$

$a_1 = 0.$

$$b_n = -\frac{\sin n\pi}{\pi(n^2-1)} \quad n>1$$

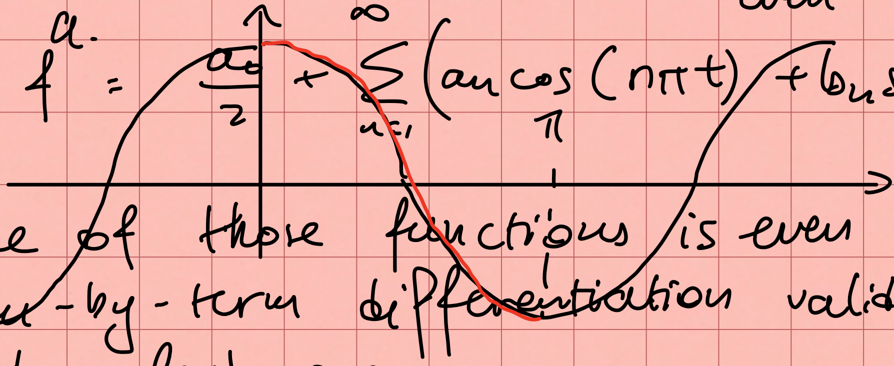
$$b_1 = \frac{1}{2}$$

Plug in into

11. a.

even extension

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi t) + b_n \sin(n\pi t) \right)$$

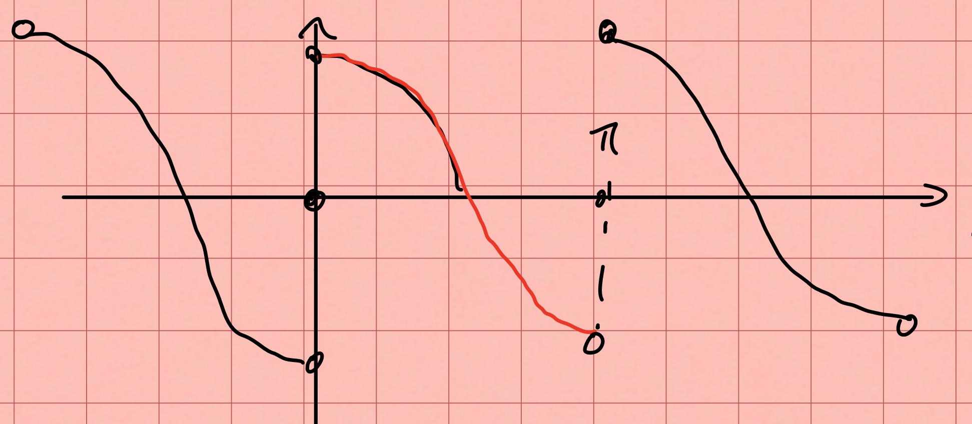


None of those functions is even or odd. Term-by-term differentiation valid only for the last one.

15 F. cosine series:

$f_1(t) = \cos(t)$

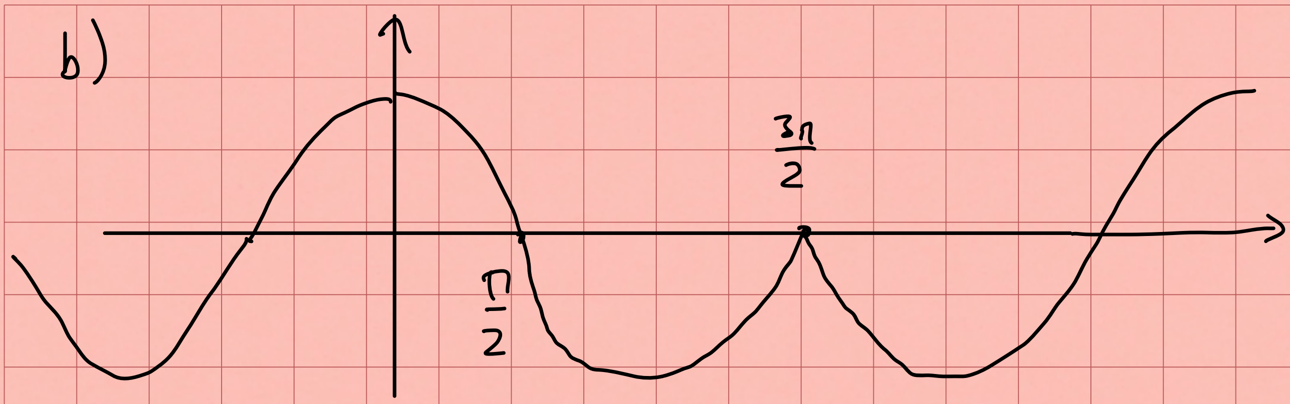
F. sine series:



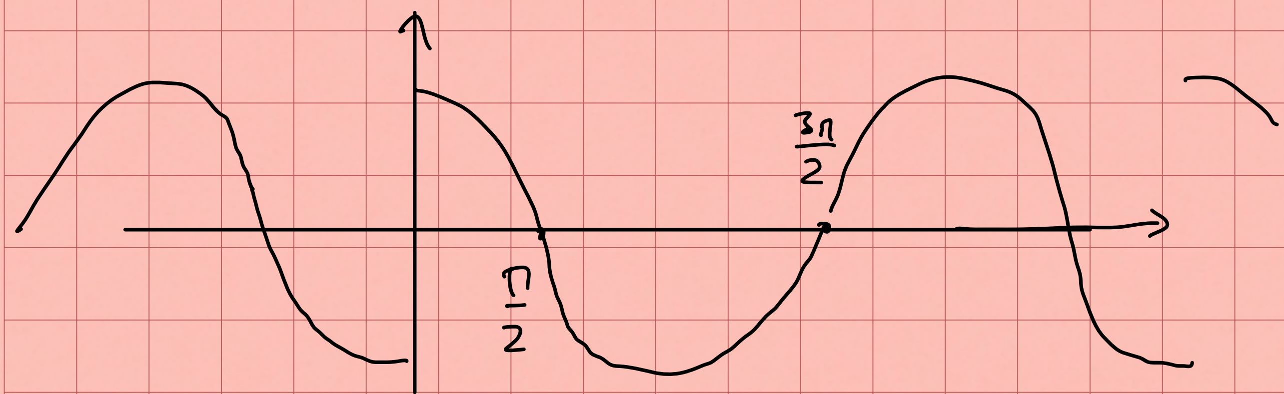
$$f = \sum_{n=2}^{\infty} \frac{2n(1+(-1)^n)}{(-1+n^2)\pi} \sin(nt)$$

notice $b_1 = 0$.

b)



$$\cos(t) = -\frac{8}{3\pi} + \sum_{n=1}^{\infty} \frac{12(-1)^n}{-9\pi+4n^2\pi} \cos\left(\frac{2n}{3} t\right)$$



$$\cos(t) = \sum_{n=1}^{\infty} \frac{8n}{-9\pi+4n^2\pi} \sin\left(\frac{2n}{3} t\right)$$

16. Write , to satisfy endpt condition

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$1 = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

Find $\begin{cases} b_n = -\dfrac{4}{\pi n(n^2+4)} & n \text{ odd} \\ b_n = 0 & n \text{ even} \end{cases}$