Parametrizing surfaces

February 27, 2018

1. **Ellipsoids:** The ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
can be parametrized by setting
$$\vec{r}(u, v) = \langle a \sin(u) \cos(v), b \sin(u) \sin(v), c \cos(u) \rangle \text{ for } (u, v) \in [0, \pi] \times [0, 2\pi].$$
As a special case, the **sphere** $x^2 + y^2 + z^2 = r^2$ can be parametrized as
$$\vec{r}(u, v) = \langle r \sin(u) \cos(v), r \sin(u) \sin(v), r \cos(u) \rangle \text{ for } (u, v) \in [0, \pi] \times [0, 2\pi].$$

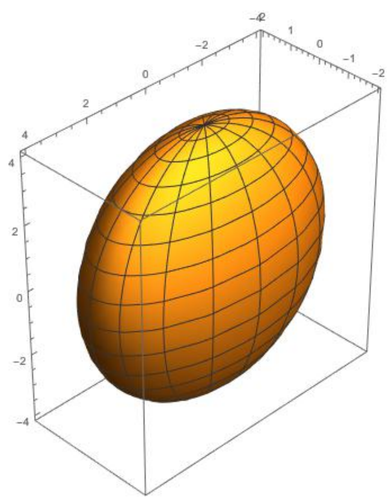

Figure 1: The ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1$

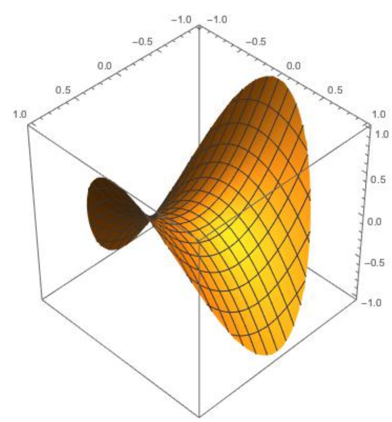

Figure 2: The graph of $f(x, y) = x^2 - y^2$, for (x, y) in the unit disk.

2. **Graphs:** When our surface is the graph of a function $f(x, y)$, where $(x, y) \in D$, we can let $x = u$, $y = v$ and $z = f(u, v)$, that is,
$$\vec{r}(u, v) = \langle u, v, f(u, v) \rangle \text{ for } (u, v) \in D.$$

3. **Cylinders:** (in a broad sense, surfaces described by an equation that only involves 2 variables, e.g. $x^2 + \dfrac{z^2}{4} = 1$.) We look at the equation of the cylinder and parametrize as if it were a plane curve, with respect to u. Then set the third variable to be v.

   In this example, we would write $x = \cos(u)$, $z = 2\sin(u)$, $u \in [0, 2\pi]$ and $y = v$, $v \in \mathbb{R}$.

4. **Surfaces of Revolution:** Say that we'd like to parametrize the surface obtained by revolving the graph of $z = f(y)$, $y \in [a, b]$ around the y axis. Then we can set $y = v$ and therefore, since for each v we have a circle of radius $f(v)$ on the plane $y = v$,

$$\vec{r}(u, v) = \langle f(v)\cos(u), v, f(v)\sin(u)\rangle, (u, v) \in [0, 2\pi] \times [a, b].$$

   We work similarly if we need to revolve about another axis.

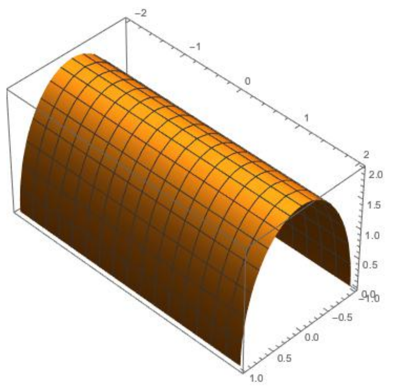

Figure 3: The cylinder $x^2 + \frac{z^2}{4} = 1$, $z \geq 0$ parametrized as $x = \cos(u), y = v, z = 2\sin(u)$, $u \in [0, \pi]$, $v \in \mathbb{R}$.

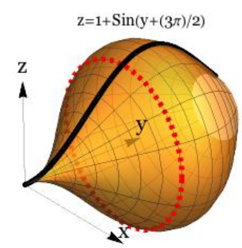

Figure 4: Revolving the curve $z = 1 + \sin(y + \frac{3\pi}{2})$ around the y axis, as $x = \left(\sin\left(v + \frac{3\pi}{2}\right) + 1\right)\cos(u)$, $y = v$ and $z = \left(\sin\left(v + \frac{3\pi}{2}\right) + 1\right)\sin(u)$.

5. **Tori:** A torus (bagel) about the z axis can be parametrized as

$$\vec{r}(u, v) = \langle (a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v)\rangle, \text{ for } (u, v) \in [0, 2\pi] \times [0, 2\pi],$$

   where $0 < b < a$.

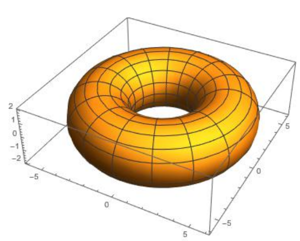

Figure 5: A torus with parametrization $x = (4 + 2\cos(v))\cos(u)$, $y = (4 + 2\cos(v))\sin(u)$, $z = 2\sin(v)$, for $(u, v) \in [0, 2\pi] \times [0, 2\pi]$.