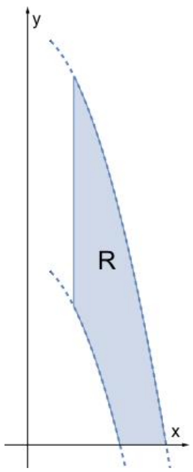
Quiz 3

Let R be the domain in the first quadrant, bounded by the curves $y = 4 - x^2$, $y = 9 - x^2$, $x = 1$ and $y = 0$. Let T be the transformation given by $x = v$, $y = u - v^2$, that is, $(x, y) = T(u, v) = (v, u - v^2)$, defined for all $(u, v) \in \mathbb{R}^2$.

1. Show that for each $(x, y) \in R$ there exists exactly one (u, v) such that $(x, y) = T(u, v)$ (in other words, show that you can solve for (u, v) if $(x, y) \in R$). This will show that there is some set S in the uv-plane such that T defined on S is invertible, and its image is R.

2. Find the set $S = T^{-1}(R)$, that is, the set of points in the uv-plane for which $T(u, v) \in R$, or equivalently, the image of R under T^{-1}.

3. Use the transformation T and your answers to compute the integral $\iint_R x\,dA$.



1. $x = v \Rightarrow v = x$

$y = u - v^2 \Rightarrow u = y + v^2 = y + x^2$

So for each (x,y) the point $(u,v) = (y+x^2, x)$ is such that $T(u,v) = (x,y)$, and $T^{-1}(xy) = (y+x^2, x)$

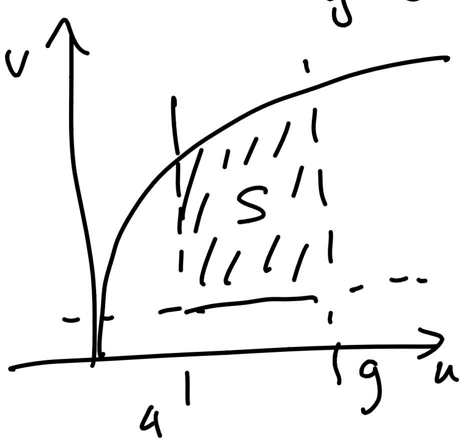
2. $y = 4 - x^2 \Rightarrow y + x^2 = 4 \Rightarrow u = 4$

$y = 9 - x^2 \Rightarrow u = 9$

$x = 1 \Rightarrow v = 1$

$y = 0 \Rightarrow u = v^2$

S is the set bounded by those curves



3. T is invertible & C^1, T^{-1} is also C^1.

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & -2v \end{vmatrix} = -1 \neq 0$$

So

$$\iint_A x\,dA = \int_4^9 \int_1^{\sqrt{u}} v\,|-1|\,dv\,du$$

$$= \int_4^9 \frac{v^2}{2}\Big|_1^{\sqrt{u}} du = \int_4^9 \frac{u-1}{2} du = \frac{u^2}{4} - \frac{1}{2}u \Big|_4^9 = \frac{81-16}{4} - \frac{1}{2}\cdot 5$$