15.7 Review

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems and understand them. Hopefully this review sheet will remind you of some of the key ideas of these sections.

15.7: Triple Integrals

1. Given a real valued three variable function $w = f(x, y, z)$ and a solid region E, we define the **triple integral** as $\iiint\limits_E f(x,y,z)\ dV = \lim\limits_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$

2. Given a solid region E, there are 6 different ways to set up our integral (one for each ordering of dx, dy and dz). So that gives a lot of options. But ultimately we group these into three major cases. **In all cases and in practice, we start by deciding what variable we want for the inner most integral.**

   (a) **Type I (TOP/BOTTOM or dz inside):**
   If $u_1(x,y) \le z \le u_2(x,y)$ and D is the projection of the solid onto the xy-plane, then
   $$\iiint\limits_E f(x,y,z)\ dV = \iint\limits_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\right)\ dA.$$

   (b) **Type II (BACK/FRONT or dx inside):**
   If $u_1(y,z) \le x \le u_2(y,z)$ and D is the projection of the solid onto the yz-plane, then
   $$\iiint\limits_E f(x,y,z)\ dV = \iint\limits_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\right)\ dA.$$

   (c) **Type III (LEFT/RIGHT or dy inside):**
   If $u_1(x,z) \le y \le u_2(x,z)$ and D is the projection of the solid onto the xz-plane, then
   $$\iiint\limits_E f(x,y,z)\ dV = \iint\limits_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right)\ dA.$$

3. So the process is:

   (a) Make a variable choice and solve each equation to get that variable by itself. (Let's say you choose the variable z)

   (b) Then draw the projection onto the other two variables. (The xy-plane if you chose z in the previous step).

      i. If the surface actually cross the plane, then you can set the 'inner' variable choice to zero ($z = 0$) in the surface to find the curve in which the surface intersect the plane.

      ii. If there are two surfaces you may need to find the curve of intersection (meaning you have z = surface 1 function and z = surface 2 function, you set surface 1 function = z = surface 2 function and simplify to get the curve of intersection).

      iii. Graph all these in the other two variables.

   (c) Then use the techniques of 15.3 and 15.4 to describe D with inequalities.

4. Applications:

$$\iiint\limits_E 1 \; dV = \text{Volume of E.}$$

$$\tfrac{1}{V(E)} \iiint\limits_E f(x,y,z) \; dV = \text{Average value of } f(x,y,z) \text{ on E.}$$

$$\begin{aligned}
m &= \iiint\limits_E \rho(x,y,z) \; dV = \text{total mass.} \\
M_{yz} &= \iiint\limits_E x\rho(x,y,z) \; dV = \text{moment } \textit{about} \text{ the } yz\text{-plane, and} \quad \bar{x} = \frac{M_{yz}}{m}. \\
M_{xz} &= \iiint\limits_E y\rho(x,y,z) \; dV = \text{moment } \textit{about} \text{ the } xz\text{-plane, and} \quad \bar{y} = \frac{M_{xz}}{m}. \\
M_{xy} &= \iiint\limits_E z\rho(x,y,z) \; dV = \text{moment } \textit{about} \text{ the } xy\text{-plane, and} \quad \bar{z} = \frac{M_{xy}}{m}. \\
Q &= \iiint\limits_E \sigma(x,y,z) \; dV = \text{total charge.}
\end{aligned}$$