16.5: Div and Curl

In this section we pick up a couple operations that give information about a vector field in $\mathbb{R}^3$. They are essential for study of $\mathbb{R}^3$, are very useful in physics, and will be important throughout the remainder of the course.

Gradient: You already know this one. The gradient operator is

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}, \quad \text{so that} \quad \nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Notice that ∇: **scalar field** $\to$ **vector field**. We have already discussed the significance of the gradient vector in 14.6.

Curl: For a vector field $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, we define:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}$$

Notice that curl: **vector field** $\to$ **vector field**.

1. For interpretation, perhaps it is best to think of a velocity vector field for a fluid. The curl $\mathbf{F}(x_0,y_0,z_0)$ gives information about the rotation in the vector field around the point (x_0,y_0,z_0). In particular, the direction of curl $\mathbf{F}$ is the axis of rotation about the point with orientation following the right hand rule. And the magnitude of curl $\mathbf{F}$ gives information about the speed of rotation (angular speed). We will discuss this more in section 16.8. If $\operatorname{curl}\mathbf{F}(x_0,y_0,z_0) = \mathbf{0}$, then we say that $\mathbf{F}$ is *irrotational* at (x_0,y_0,z_0). And the fact below says that a conservative vector field is irrotational everywhere.

2. An important fact was: If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ and the components have continuous partials, then

$$\mathbf{F} \text{ is conservative if and only if } \operatorname{curl}\mathbf{F} = \mathbf{0}.$$

   Hence we have a test to determine if a vector field in $\mathbb{R}^3$ is conservative (remember in $\mathbb{R}^2$ the test is $P_y = Q_x$ which is a special case of this more general fact).

Divergence: For a vector field $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, we define:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that div: **vector field** $\to$ **scalar field**.

1. For interpretation, again it is best to think of a velocity vector field for a fluid. The div $\mathbf{F}(x_0,y_0,z_0)$ gives information about the tendency of a particle to diverge, move away from, (x_0,y_0,z_0). In a fluid example it typically measures the net mass per volume per time (the net rate of change of mass per volume). That is, it measures how 'outgoing' the vector field is at (x_0,y_0,z_0). If $\operatorname{div}\mathbf{F}(x_0,y_0,z_0) = 0$, then we say that $\mathbf{F}$ is *incompressible* at (x_0,y_0,z_0). We will discuss this more in 16.9.

2. An interesting note was: For any vector field $\mathbf{F}$ on $\mathbb{R}^3$ whose components have continuous second partials

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0.$$

Hence if $\operatorname{div}\mathbf{G} \neq 0$ for some vector field $\mathbf{G}$, then we would know that $\mathbf{G} \neq \operatorname{curl}\mathbf{F}$ for any vector field $\mathbf{F}$ (in other words some vector fields cannot possible be the curl of some other vector field).

Laplacian: For a scalar field $f(x, y, z)$, we make a special note (this won't come up much in this class, but you will see it in differential equations and physics):

$$\operatorname{div}\nabla f = \nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Notice that ∇^2: **scalar field** $\to$ **scalar field**.

1. This is just a special case where we are computing the divergence of a conservative vector field. So the interpretation is as it was for divergence. This measurement of divergence (or flux) from a point in a conservative vector field is of fundamental importance in many areas of physics (thermodynamics, diffusion, minimizing energy, and more). And it comes up a lot as you discuss motion and differential equations in $\mathbf{R}^3$.

We ended the section with two vector forms of Green's Theorem (here $\mathbf{T}$ is the unit tangent and $\mathbf{n}$ is the outward pointing unit normal). Here $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{i} + 0\mathbf{k}$.

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D (\operatorname{curl}\mathbf{F}) \cdot \mathbf{k} dA$$

and

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \oint_C -Q\,dx + P\,dy = \iint_D \operatorname{div}\mathbf{F} dA.$$