Journal of Mathematics Research; Vol. 14, No. 4; August 2022
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Correlation Functions of Quantum Toroidal $\mathfrak{gl}_n$ Algebra

Hao Cui

Correspondence: Marc Garneau Collegiate Institute, Ontario, Canada

Received: June 12, 2022 Accepted: July 13, 2022 Online Published: July 26, 2022

doi:10.5539/jmr.v14n4p94 URL: https://doi.org/10.5539/jmr.v14n4p94

Abstract

In this paper, we study the correlation functions of the quantum toroidal $\mathfrak{gl}_n$ algebra. Their first key properties are established in analogy to those of the correlation functions of quantum affine algebras $U_q\mathfrak{n}_+$. The core of the paper is the proof of the vanishing of those correlation functions at length 3 wheel conditions, which is done separately for $n = 1, n = 2, n \geq 3$ cases with different "Master Equalities" of formal series (the $n = 1$ case has been previously discussed by the author in (Cui, 2021)). Another important contribution is the discovery of cubic Serre relations for the quantum toroidal.

1. Introduction

The quantum toroidal algebras of $\mathfrak{gl}_n$, where $n > 2$, were introduced more than 20 years ago in (Ginzburg, 1995). However, their representations are not fully understood yet. Surprisingly, the $\mathfrak{gl}_1$ counterpart of those algebras, the quantum toroidal $\mathfrak{gl}_1$, has been introduced much later in (Feigin, 2011) and (Miki, 2007) and has attracted lots of attention over the last decade from both mathematicians and physicists, due to its close relation to several other topics, which include

- the q-AGT conjecture
- spherical DAHA
- knot invariants
- the Hall algebra of an elliptic curve

Finally, the proper definition of the quantum toroidal algebra of $\mathfrak{gl}_2$ was given only recently in (Feigin, 2016).

Yet another combinatorial perspective to the quantum toroidal algebras is given via the trigonometric version of the Feigin-Odesskii elliptic shuffle algebras of (Feigin, 1997). In this approach, the elements of the "positive part" of the quantum toroidal algebra are realized as rational symmetric functions subject to rather simple "pole condition" and more interesting "wheel condition," which specify the vanishing of those functions under certain specializations of the variables to a multiple of each other.

An interesting perspective to the aforementioned "wheel" conditions was provided by Enriquez in (Enriquez, 2000), where he explained how these conditions arise naturally in the study of the so-called correlation functions of quantum affine algebras. The main objective of this paper is to establish similar properties of the correlation functions of the quantum toroidal algebras, thus providing yet another perspective to the "wheel" conditions in the present setup. The case of quantum toroidal algebras is particularly interesting in the setup of $\mathfrak{gl}_n$, since this is the only case when one has two deformation parameters instead of a single one. Historically, it is common to encode those two parameters via q_1, q_2, q_3 subject to the equality $q_1q_2q_3 = 1$.

However, for $n > 2$, the defining relations for the quantum toroidal $\mathfrak{gl}_n$ algebras look very similar to those of the quantum affine algebras of $\mathfrak{gl}_n$, which will allow us to deduce the corresponding vanishing property from that of (Cui, 2021). This only leaves the case $n = 2$ to consider, (the case $n = 1$ was treated in (Cui, 2021)) which is the primary subject of the present note. In this case, we hook out the wheel condition by establishing a "Master Equality" of formal series. However, the interesting discovery is that although the Master Equalities look vastly different for the cases $n = 1, n = 2$ and $n \geq 3$, the wheel conditions for all quantum toroidal $\mathfrak{gl}_n$ algebras are quite similar in nature.

In addition, as we shall see, the studies of the correlation function for the quantum toroidal $\mathfrak{gl}_2$ poses a particular problem. When the Master Equalities were established in (Enriquez, 2000) and for the case of quantum toroidal $\mathfrak{gl}_n$, the "Serre relation" that was essential in the process are of degree 3. However, the cubic form of the Serre relation for the quantum toroidal $\mathfrak{gl}_2$ case has not been discussed in the literature before. Thus in order to hook out the Wheel condition, the cubic Serre relations for the quantum toroidal algebra of $\mathfrak{gl}_2$ has to be determined first.

2. Quantum Toroidal $\mathfrak{gl}_n (n \geq 3)$

2.1 General Setup

Given non-zero complex numbers $q_1, q_2, q_3 \in \mathbb{C}^\times$ satisfying $q_1q_2q_3 = 1$, the quantum toroidal algebra $U_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ with $n \geq 3$ is defined uniquely. The algebra has generators $\{e_{i,k}, f_{i,k}, \psi^{\pm}_{i,s}\}$, where $k \in \mathbb{Z}$, $s \geq 0$, and $i \in \mathbb{Z}_n$ viewed as residues modulo n. More often, the numbers q_1, q_2, q_3 will be written as $\frac{d}{q}, q^2, \frac{1}{dq}$ respectively for some complex numbers $d, q \in \mathbb{C}^\times$.

These algebras have the so-called "triangular decomposition", the first part of which means that the multiplication map induces the vector space isomorphism:

$$U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n) \otimes U^0_{q_1,q_2,q_3}(\mathfrak{gl}_n) \otimes U^-_{q_1,q_2,q_3}(\mathfrak{gl}_n) \xrightarrow{\sim} U_{q_1,q_2,q_3}(\mathfrak{gl}_n) \tag{1}$$

where $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n), U^0_{q_1,q_2,q_3}(\mathfrak{gl}_n), U^-_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ are subalgebras generated by $\{e_{i,k}\}, \{\psi^{\pm}_{i,s}\}, \{f_{i,k}\}$ respectively, while the second part of the triangular decomposition is the fact that each of these subalgebras is defined by the above generators and the corresponding relations.

Remark. *We note that the entire quantum toroidal algebra $U_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ can be recovered as a* **Drinfeld double** *of $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n)$.*

Explicitly, the "positive" subalgebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ is generated by $\{e_{i,k}\}^{k\in\mathbb{Z}}_{i\in\mathbb{Z}_n}$ subject to the quadratic relations and cubic/quadratic Serre relations specified below:

Definition 1. *The quadratic relation takes the following form:*

$$d_{ij}g_{ij}(z,w)e_i(z)e_j(w) = -g_{ji}(w,z)e_j(w)e_i(z). \tag{2}$$

where the series $e_i(z)$ is the formal series defined via

$$e_i(z) := \sum_{k\in\mathbb{Z}} e_{i,k}z^{-k}$$

which represents the current of "color i". Here, the constant d_{ij} is defined as follows:

$$d_{ij} = \begin{cases} d^{-1} & \text{if } j = i+1 \\ d & \text{if } j = i-1 \\ 1 & \text{otherwise} \end{cases}$$

while

$$g_{ij}(z,w) = z - q^{a_{ij}}d^{-m_{ij}}w, \tag{3}$$

with

$$a_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}, \qquad m_{ij} = \delta_{i,j+1} - \delta_{i,j-1}, \tag{4}$$

for $i, j \in \mathbb{Z}_n$, where δ_{ij} is the Kronecker Delta.

Definition 2. *Given any function $F(a,b)$, in 2 variables, its "symmetrization"*

$$\sum_{sym\{a,b\}} F(a,b)$$

is defined via

$$\sum_{sym\{a,b\}} F(a,b) := F(a,b) + F(b,a)$$

Definition 3. *The cubic Serre relation for these algebras takes the following form:*

$$\sum_{sym\{z_1,z_2\}} [e_i(z_1), [e_i(z_2), e_{i\pm1}(w)]_q]_{q^{-1}} = 0, \tag{5}$$

where $[a,b]_q = ab - q\cdot ba$. In addition, if $j \neq i, i-1, i+1$, then we also impose quadratic Serre relation:

$$[e_i(z), e_j(w)] = 0. \tag{6}$$

There are two crucial series (7), (8) that we shall be using in the rest of the paper.

Definition 4. *For the formal variables x, y, the* **delta-function** *series $\delta(x, y)$ is defined via*

$$\delta(x, y) = \sum_{k\in\mathbb{Z}} x^k y^{-k-1}. \tag{7}$$

Definition 5. *For the formal variables x, y, the series $\frac{1}{x-y}$ is defined via*

$$\frac{1}{x-y} := \frac{1}{x}\sum_{i=0}^{\infty}\left(\frac{y}{x}\right)^i = \sum_{i<0} x^i y^{-i-1}. \tag{8}$$

Note that the series in (8) converges to $\frac{1}{x-y}$ in the region $|y| < |x|$ of $\mathbb{C}^2$.

Remark. *We note the following equivalent expression for the delta-function:*

$$\delta(x, y) = \frac{1}{x-y} + \frac{1}{y-x}. \tag{9}$$

2.2 Correlation Functions and Their Properties

Similar to what the author did in (Cui, 2021), let V be a *highest weight* representation of the algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n)$, and its correlation function is defined as below:

Definition 6. *For any vector $v \in V$, and a covector $\epsilon \in V^*$(the dual of V), consider the* ***correlation function*** *of the algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ defined via*

$$f\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right) = \left\langle \epsilon, e_{i_1}\left(z_1^{(i_1)}\right)\cdots e_{i_N}\left(z_N^{(i_N)}\right) v\right\rangle, \tag{10}$$

where i_k is the $\mathbb{Z}_n$-"color" of the formal variable z_k.

Our main result is that the correlation functions have to take a certain form:

Theorem 7.

$$f\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right) = \frac{\prod_{a<b}^{i_a=i_b}\left(z_a^{(i_a)} - z_b^{(i_b)}\right) \times A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)}{\prod_{a<b} g_{i_a i_b}\left(z_a^{(i_a)}, z_b^{(i_b)}\right)}. \tag{11}$$

In the equation above, $A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)$ is a color-symmetric Laurent polynomial and is subject to a certain "Vanishing condition" discussed below.

Note that the theorem above, besides the vanishing condition, easily follows from the same deduction as in case of $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, which has been demonstrated by the author in (Cui, 2021). Thus, the proof can be omitted.

Now, let us describe the vanishing properties for the function $A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)$ in this case.

Theorem 8 (Vanishing property)**.** *If there exists 3 distinct variables $\left\{z_p^{(i_p)}, z_q^{(i_q)}, z_r^{(i_r)}\right\}$, where one of the following two conditions are satisfied:*

1. $i_p = i_q, i_r = i_p + 1, z_p^{(i_p)} = t, z_q^{(i_q)} = q^{-2}t, z_r^{(i_r)} = dq^{-1}t$
2. $i_p = i_q, i_r = i_p - 1, z_p^{(i_p)} = t, z_q^{(i_q)} = q^{-2}t, z_r^{(i_r)} = d^{-1}q^{-1}t$

then the laurant polynomial $A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)$ vanishes.

Instead of proving this result from scratch, let us deduce it from a similar result for quantum affine $\mathfrak{sl}_3$ established in (Enriquez, 2000).

Note that the quadratic relation in (2) implies two identities:

$$\begin{aligned}(z - q^2w)e_i(z)e_i(w) &= (q^2z - w)e_i(w)e_i(z)\\ (d^{-1}z - q^{-1}w)e_i(z)e_{i+1}(w) &= (q^{-1}d^{-1}z - w)e_{i+1}(w)e_i(z).\end{aligned} \tag{12}$$

Now, we are going to introduce some new notations. Define $\widetilde{e}_i(z), \widetilde{e}_{i+1}(w)$ via

$$\widetilde{e}_i(z) = e_i(z), \qquad \widetilde{e}_{i+1}(w) = e_{i+1}(\lambda w) \tag{13}$$

for some λ to be determined later. Now, we shall replace the e-currents in the quadratic relations (12) with the $\widetilde{e}$-currents. Note that the first relation in (12) stays the same under such a replacement. After the substitution takes place in the second equality, we get

$$\widetilde{e}_i(z)\widetilde{e}_{i+1}(w) = \widetilde{e}_{i+1}(w)\widetilde{e}_i(z) \cdot \frac{q^{-1}d^{-1}z - \lambda w}{d^{-1}z - q^{-1}\lambda w} \tag{14}$$

Note that when we choose $\lambda = d^{-1}$, the last factor in the RHS of (14) is equal to $\frac{q^{-1}z-w}{z-q^{-1}w}$. And that factor is exactly the same as the factor involved in the quadratic relation for the currents of the quantum affine $\mathfrak{sl}_3$ algebra studied in (Enriquez, 2000). Thus, according to (Enriquez, 2000), we have:

Lemma 9. *The function $A\left(\widetilde{z_1}^{(i_1)}, \cdots, \widetilde{z_N}^{(i_N)}\right)$ vanishes when there exists 3 variables $\widetilde{z_a}^{(i_a)}, \widetilde{z_b}^{(i_b)}, \widetilde{z_c}^{(i_c)}$, such that $\widetilde{z_a}^{(i_a)}, \widetilde{z_b}^{(i_b)}$ are are associated to the current $\widetilde{e}_i$ and $\widetilde{z_c}^{(i_c)}$ is associated to the current $\widetilde{e}_{i+1}$ and furthermore $\widetilde{z_a}^{(i_a)} = t, \widetilde{z_b}^{(i_b)} = q^{-2}t$ and $\widetilde{z_c}^{(i_c)} = q^{-1}t$.*

Switching back from $\{\widetilde{z_a}^{(i_a)}, \widetilde{z_b}^{(i_b)}, \widetilde{z_c}^{(i_c)}\}$ into $\{z_a^{(i_a)}, z_b^{(i_b)}, z_c^{(i_c)}\}$, where $z_a^{(i_a)}, z_b^{(i_b)}$ are associated with the current e_i and $z_c^{(i_c)}$ is associated with the current e_{i+1}. We see that Lemma 9 implies the first condition in Theorem 8.

To see the second condition of Theorem 8, one needs to follow a similar argument but considering $e_{i-1}(w)$ instead of $e_{i+1}(w)$. The result now follows from that of (Enriquez, 2000) by considering $\widetilde{e}_i(z) = e_i(z), \widetilde{e}_{i-1}(w) = e_{i-1}(\lambda w)$, where $\lambda = d$.

3. Quantum Toroidal $\mathfrak{gl}_2$

3.1 General Setup

Let us now consider the remaining case of the quantum toroidal algebra of $\mathfrak{gl}_2$. The algebra $U_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ is generated by $\{e_{i,k}, f_{i,k}, \psi^{\pm}_{i,s}\}$ with $i \in \mathbb{Z}_2, k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}$ subject to their respective defining relations.

Definition 10. *The quadratic relations take the following form:*

$$\begin{gathered}
(z - q^2 w)e_i(z)e_i(w) = (q^2 z - w)e_i(w)e_i(z) \\
(q^2 z - w)f_i(z)f_i(w) = (z - q^2 w)f_i(w)f_i(z) \\
(z - dq^{-1}w)(z - (dq)^{-1}w)e_i(z)e_{i+1}(w) = (w - dq^{-1}z)(w - (dq)^{-1}z)e_{i+1}(w)e_i(z) \\
(w - dq^{-1}z)(w - (dq)^{-1}z)f_i(z)f_{i+1}(w) = (z - dq^{-1}w)(z - (dq)^{-1}w)f_{i+1}(w)f_i(z) \\
(z - q^2 w)\psi^{\pm}_i(z)e_i(w) = (q^2 z - w)e_i(w)\psi^{\pm}_i(z) \\
\psi^{\pm}_i(z)\psi^{\mp}_j(w) = \psi^{\mp}_j(w)\psi^{\pm}_i(z) \\
(z - dq^{-1}w)(z - (dq)^{-1}w)\psi^{\pm}_i(z)e_{i+1}(w) = (w - dq^{-1}z)(w - (dq)^{-1}z)e_{i+1}(w)\psi^{\pm}_i(z) \\
(q^2 z - w)\psi^{\pm}_i(z)f_i(w) = (z - q^2 w)f_i(w)\psi^{\pm}_i(z) \\
(w - dq^{-1}z)(w - (dq)^{-1}z)\psi^{\pm}_i(z)f_{i+1}(w) = (z - dq^{-1}w)(z - (dq)^{-1}w)f_{i+1}(w)\psi^{\pm}_i(z) \\
[e_i(z), f_{i+1}(w)] = 0 \\
[e_i(z), f_{i+1}(w)] = (q - q^{-1})^{-1}(\psi^+_i(z) - \psi^-_i(z)) \cdot \delta\left(\frac{z}{w}\right)
\end{gathered} \tag{15}$$

Definition 11 (Quartic Serre)**.** *The quartic Serre relation takes the following form:*

$$\sum_{sym\{z_1,z_2,z_3\}} [e_i(z_1)[e_i(z_2), [e_i(z_3), e_{i+1}(w)]_{q^2}]_1]_{q^{-2}} = 0. \tag{16}$$

and the relations for f-currents are also similar.

Note that after expanding the q-brackets, the quartic general Serre relation is equivalent to

$$\begin{aligned}
\sum_{sym\{z_1,z_2,z_3\}} \Big(& e_i(z_1)e_i(z_2)e_i(z_3)e_j(w) - pe_i(z_1)e_i(z_2)e_j(w)e_i(z_3) \\
& + pe_i(z_1)e_j(w)e_i(z_2)e_i(z_3) - e_j(w)e_i(z_1)e_i(z_2)e_i(z_3)\Big) = 0
\end{aligned} \tag{17}$$

where $p = 1 + q^2 + q^{-2}$.

Let $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$, $U^0_{q_1,q_2,q_3}(\mathfrak{gl}_2)$, $U^-_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ be the subalgebras generated by $\{e_{i,k}\}, \{f_{i,k}\}, \{\psi^\pm_{i,r}\}$. Then, the first part of the triangular decomposition statement holds, i.e. the multiplication map

$$U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2) \otimes U^0_{q_1,q_2,q_3}(\mathfrak{gl}_2) \otimes U^-_{q_1,q_2,q_3}(\mathfrak{gl}_2) \to U_{q_1,q_2,q_3}(\mathfrak{gl}_2) \tag{18}$$

is a vector space isomorphism.

However, the second part of the triangular decomposition claim fails, that is, $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ and $U^-_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ are the non-trivial quotients of the algebras generated by $\{e_{i,k}\}, \{f_{i,k}\}$ subject only to (17). Thus, one must first recover the new cubic Serre relation in order to make both parts of the triangular decomposition statement to hold.

Similar to Section 2, we will primarily consider the algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ generated by $\{e_{i,k}\}^{k\in\mathbb{Z}}_{i\in\mathbb{Z}_2}$ because of the idea of triangular decompositions. However, as we will see later in the present section, the currents of $U^0_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ and $U^-_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ are essential in the derivation of the cubic Serre relation.

3.2 New Cubic Serre Relations

To obtain the cubic Serre relation from the general quartic Serre relation, we will have to first look at the simplest case of the general relation. Specifically, by looking at the coefficient of $z_1^0 z_2^0 z_3^0 w^{-k}$ for some $k \in \mathbb{Z}$ in (17), we get the following identity:

$$e^3_{i,0}e_{j,k} - pe^2_{i,0}e_{j,k}e_{i,0} + pe_{i,0}e_{j,k}e^2_{i,0} - e_{j,k}e^3_{i,0} = 0 \qquad \text{for } \ j \neq i \tag{19}$$

where $p = 1 + q^2 + q^{-2}$ as before.

The first non-trivial step is to commute the equation above with the $f_{i,k}$ generators.

Remark. *Commuting* (19) *with $f_{i,0}$ results in the trivial equality.*

Thus, we shall instead commute it with $(q - q^{-1})f_{i,1}$.

Lemma 12. *The following two identities hold:*

$$\begin{aligned} &[e_{i,0}, (q-q^{-1})f_{i,1}] = \psi^+_{i,1} \\ &[e_{j,k}, (q-q^{-1})f_{i,1}] = 0 \quad \textit{for} \ \ j \neq i \end{aligned} \tag{20}$$

Theorem 13. *When we commute (19) with $(q - q^{-1})f_{i,1}$, we get*

$$\begin{aligned} &\left(\psi^+_{i,1}e^2_{i,0} + e_{i,0}\psi^+_{i,1}e_{i,0} + e^2_{i,0}\psi^+_{i,1}\right)e_{j,k} - p\left(\psi^+_{i,1}e_{i,0}e_{j,k}e_{i,0} + e_{i,0}\psi^+_{i,1}e_{j,k}e_{i,0} + e^2_{i,0}e_{j,k}\psi^+_{i,1}\right) \\ &+p\left(\psi^+_{i,1}e_{j,k}e^2_{i,0} + e_{i,0}e_{j,k}\psi^+_{i,1}e_{i,0} + e_{i,0}e_{j,k}e_{i,0}\psi^+_{i,1}\right) - e_{j,k}\left(\psi^+_{i,1}e^2_{i,0} + e_{i,0}\psi^+_{i,1}e_{i,0} + e^2_{i,0}\psi^+_{i,1}\right) = 0 \end{aligned} \tag{21}$$

Next, let us rewrite (21) by pulling all the ψ terms to the leftmost side in each term. For the sake of simplicity, we shall only demonstrate how to do so for the first term in (21).

According to the quadratic relation, we have that:

$$(z - q^2w)\psi^+_i(z)e_i(w) = (q^2z - w)e_i(w)\psi^+_i(z). \tag{22}$$

which implies the following:

Lemma 14. *The following identity holds by checking the coefficients in the quadratic relation:*

$$\psi^+_{i,k+1}e_{i,l} - q^2\psi^+_{i,k}e_{i,l+1} = q^2e_{i,l}\psi^+_{i,k+1} - e_{i,l+1}\psi^+_{i,k} \tag{23}$$

In (23), if $k = -1$, then $\psi^+_{i,-1} = 0$ and we recover that

$$\psi^+_{i,0}e_{i,l} = q^2e_{i,l}\psi^+_{i,0}. \tag{24}$$

Moreover, if $k, l = 0$, then we get:

$$\begin{aligned} \psi^+_{i,1}e_{i,0} - q^2\psi^+_{i,0}e_{i,1} &= q^2 e_{i,0}\psi^+_{i,1} - e_{i,1}\psi^+_{i,0} \\ \implies q^2 e_{i,0}\psi^+_{i,1} &= \psi^+_{i,1}e_{i,0} + e_{i,1}\psi^+_{i,0} - q^2\psi^+_{i,0}e_{i,1} \\ \implies e_{i,0}\psi^+_{i,1} &= q^{-2}\psi^+_{i,1}e_{i,0} + (q^{-4}-1)\psi^+_{i,0}e_{i,1} \end{aligned} \tag{25}$$

We shall also need the following particular case of the first quadratic relation of (15) obtained by comparing the coefficients of z^0w^0:

$$e_{i,1}e_{i,0} = q^2 e_{i,0}e_{i,1} \tag{26}$$

Due to (23), (24) and (25), we can finally start operating on the first sum in (21).

Theorem 15. *The first sum in* (21) *equals*

$$(1 + q^{-2} + q^{-4})\psi^+_{i,1}e^2_{i,0}e_{j,k} + (1 + q^{-2})(q^{-4} - q^2)\psi^+_{i,0}e_{i,0}e_{i,1}e_{j,k} \tag{27}$$

By following a similar strategy, let us also pull ψ-terms to the left in the other 3 terms of (21). We note the following lemma:

Lemma 16. *The following identity follows from* (15)*:*

$$e_{j,l}\psi^+_{i,1} = q^2\psi^+_{i,1}e_{j,l} + (q^2-1)(qd^{-1}+qd)\psi^+_{i,0}e_{j,l+1} \quad \text{for } j \neq i, l \in \mathbb{Z}. \tag{28}$$

The proof is completely analogous to that of Lemma 14. Thus, we obtain:

Theorem 17. *The second term in* (21) *equals*

$$\begin{aligned} &\psi^+_{i,1}e_{i,0}e_{j,k}e_{i,0} + e_{i,0}\psi^+_{i,1}e_{j,k}e_{i,0} + e^2_{i,0}e_{j,k}\psi^+_{i,1} \\ =&\psi^+_{i,1}\Big((1+q^{-2})e_{i,0}e_{j,k}e_{i,0} + q^{-2}e^2_{i,0}e_{j,k}\Big) \\ +&\psi^+_{i,0}\Big((q^{-4}-1)e_{i,1}e_{j,k}e_{i,0} + (q^{-2}-q^{-4})(qd^{-1}+qd)e^2_{i,0}e_{j,k+1} + (q^2+1)(q^{-4}-1)e_{i,0}e_{i,1}e_{j,k}\Big) \end{aligned} \tag{29}$$

The third term in (21) *equals*

$$\begin{aligned} &\psi^+_{i,1}e_{j,k}e^2_{i,0} + e_{i,0}e_{j,k}\psi^+_{i,1}e_{i,0} + e_{i,0}e_{j,k}e_{i,0}\psi^+_{i,1} \\ =&\psi^+_{i,1}\Big(e_{j,k}e^2_{i,0} + (q^{-2}+1)e_{i,0}e_{j,k}e_{i,0}\Big) \\ +&\psi^+_{i,0}\Big((q^2+1)(q^{-4}-1)e_{i,1}e_{j,k}e_{i,0} + (1-q^{-4})(qd^{-1}+qd)e_{i,0}e_{j,k+1}e_{i,0} + (q^{-4}-1)e_{i,0}e_{j,k}e_{i,1}\Big) \end{aligned} \tag{30}$$

And the last term in (21) *equals*

$$\begin{aligned} &e_{j,k}\Big(\psi^+_{i,1}e^2_{i,0} + e_{i,0}\psi^+_{i,1}e_{i,0} + e^2_{i,0}\psi^+_{i,1}\Big) \\ =&\psi^+_{i,1}(q^2+1+q^{-2})e_{j,k}e^2_{i,0} \\ +&\psi^+_{i,0}\Big((q^2+1+q^{-2})(1-q^{-2})(qd^{-1}+qd)e_{j,k+1}e^2_{i,0} + (q^2+1+q^{-2})(q^{-2}-q^2)e_{j,k}e_{i,0}e_{i,1}\Big) \end{aligned} \tag{31}$$

Now, we know that the 4 equations declared above add up to 0 due to (21). Thus, we would group terms with the same ψ-term in front. However, in that process, it can be easily checked that the terms with $\psi^+_{i,1}$ cancel out, and we only have to deal with the terms with $\psi^+_{i,0}$.

Lemma 18. *The following expression contains the sum of all terms with* $\psi^+_{i,0}$

$$\begin{aligned} &\psi^+_{i,0}(q^2+1+q^{-2})(1-q^{-2})\times \\ &\Big(-q^{-2}(qd^{-1}+qd)e^2_{i,0}e_{j,k+1} + (1+q^{-2})(qd^{-1}+qd)e_{i,0}e_{j,k+1}e_{i,0} - (qd^{-1}+qd)e_{j,k+1}e^2_{i,0} \\ &+ (1+q^2)e_{i,0}e_{i,1}e_{j,k} + (1+q^2)e_{j,k}e_{i,0}e_{i,1} - (1+q^{-2})e_{i,0}e_{j,k}e_{i,1} - (1+q^2)e_{i,1}e_{j,k}e_{i,0}\Big) \end{aligned} \tag{32}$$

In particular, this sum must vanish.

However, it can be easily checked that we can simplify it to the following form:

Proposition 19. *The fact that (32) equals 0 is equivalent to the following identity:*

$$(1+q^2)[[e_{j,k}, e_{i,0}]_{q^{-2}}, e_{i,1}]_1 = (qd^{-1}+qd)[[e_{j,k+1}, e_{i,0}]_{q^{-2}}, e_{i,0}]_1 \tag{33}$$

Thus, (33) already provides a specific cubic Serre relation. However, we would like to get the more general relation, i.e. cubic relations with arbitrary modes allowed instead of $e_{i,0}, e_{i,1}$. Such generaliziation utilizes the following important elements $h_{i,s}$:

Proposition 20. *For $s \neq 0$, there exist an element $h_{i,s} \in U^0_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ such that*

$$\begin{aligned} &[h_{i,s}, e_{j,k}] = 0 \quad \text{for } j \neq i \text{ and any } k \in \mathbb{Z} \\ &[h_{i,s}, e_{i,k}] = e_{i,k+s} \end{aligned} \tag{34}$$

Before presenting specific formulas for such h-terms, let us first illustrate how they allow us to deduce general cubic Serre relations from Lemma 19.

Lemma 21. *Fix any $l_1, l_2 \in \mathbb{Z}$. Commuting (33) with h_{i,l_1} and then h_{i,l_2} gives:*

$$\begin{aligned} &(1+q^2)\Big([[e_{j,k}, e_{i,l_1+l_2}]_{q^{-2}}, e_{i,1}]_1 + [[e_{j,k}, e_{i,l_1}]_{q^{-2}}, e_{i,l_2+1}]_1 \\ &+[[e_{j,k}, e_{i,l_2}]_{q^{-2}}, e_{i,l_1+1}]_1 + [[e_{j,k}, e_{i,0}]_{q^{-2}}, e_{i,l_1+l_2+1}]_1\Big) \\ &=(qd^{-1}+qd)\Big([[e_{j,k+1}, e_{i,l_1+l_2}]_{q^{-2}}, e_{i,0}]_1 + [[e_{j,k+1}, e_{i,l_1}]_{q^{-2}}, e_{i,l_2}]_1 \\ &+[[e_{j,k+1}, e_{i,l_2}]_{q^{-2}}, e_{i,l_1}]_1 + [[e_{j,k+1}, e_{i,0}]_{q^{-2}}, e_{i,l_1+l_2}]_1\Big) \end{aligned} \tag{35}$$

Lemma 22. *Commmuting (33) with h_{i,l_1+l_2} instead, we get:*

$$\begin{aligned} &(1+q^2)\Big([[e_{j,k}, e_{i,l_1+l_2}]_{q^{-2}}, e_{i,1}]_1 + [[e_{j,k}, e_{i,0}]_{q^{-2}}, e_{i,l_1+l_2+1}]_1\Big) \\ &=(qd^{-1}+qd)\Big([[e_{j,k+1}, e_{i,l_1+l_2}]_{q^{-2}}, e_{i,0}]_1 + [[e_{j,k+1}, e_{i,0}]_{q^{-2}}, e_{i,l_1+l_2}]_1\Big) \end{aligned} \tag{36}$$

Subtracting (36) from (35), we eventually get:

$$\begin{aligned} &(1+q^2)\Big([[e_{j,k}, e_{i,l_1}]_{q^{-2}}, e_{i,l_2+1}]_1 + [[e_{j,k}, e_{i,l_2}]_{q^{-2}}, e_{i,l_1+1}]_1\Big) \\ &=(qd^{-1}+qd)\Big([[e_{j,k+1}, e_{i,l_1}]_{q^{-2}}, e_{i,l_2}]_1 + [[e_{j,k+1}, e_{i,l_2}]_{q^{-2}}, e_{i,l_1}]_1\Big) \end{aligned} \tag{37}$$

for any modes $k, l_1, l_2 \in \mathbb{Z}$. This is the general cubic Serre relation. Let us rewrite it using currents:

Theorem 23 (General Cubic Serre)**.** *The general cubic Serre relation takes the following form*

$$(1+q^2)\sum_{sym\{z_1,z_2\}}\Big([[e_j(w), e_i(z_1)]_{q^{-2}}, e_i(z_2)]_1 \cdot z_2\Big) = (qd^{-1}+qd)\sum_{sym\{z_1,z_2\}}\Big([[e_j(w), e_i(z_1)]_{q^{-2}}, e_i(z_2)]_1 \cdot w\Big) \tag{38}$$

Note that after expansion of q-brackets, we obtain the following equality:

Theorem 24.

$$\begin{aligned} \sum_{sym\{z_1,z_2\}}\Big(&e_i(z_1)e_i(z_2)e_j(w)((1+q^{-2})z_1 - (q_1+q_3)w) \\ &-e_i(z_1)e_j(w)e_i(z_2)((1+q^2)z_1 - (q_1+q_3+q_1^{-1}+q_3^{-1})w + (1+q^{-2})z_2) \\ &+e_j(w)e_i(z_1)e_i(z_2)((1+q^2)z_2 - (q_1^{-1}+q_3^{-1})w)\Big) = 0 \end{aligned} \tag{39}$$

To complete the above argument, we finally prove the existence of the h-terms, see Proposition 20.

Let

$$F(z,w) := \log(1-q^{-2}wz^{-1}) - \log(1-q^2wz^{-1}).$$

Lemma 25. *Consider elements $\{H_{i,t}\}_{t>0}$ that commute with $\{e_{i,s}\}_{s\in\mathbb{Z}}$ via*

$$[H_{i,t}, e_{i,s}] = \frac{q^{2t}-q^{-2t}}{t} e_{i,s+t}. \tag{40}$$

Then

$$\exp H_i(z)\cdot e_i(w) = e_i(w)\cdot \exp H_i(z)\cdot \exp F(z,w) \tag{41}$$

where $H_i(z) := \sum_{t>0} H_{i,t}z^{-t}$

Proof. We shall ignore the index i for simplicity. Using (40) let us then compute the commutation between $H(z)$ and $e(w)$:

$$\begin{aligned}
[H(z), e(w)] &= \sum_{t>0, s\in\mathbb{Z}} z^{-t}w^{-s}[H_t, e_s] \\
&= \sum_{t>0, r=s+t\in\mathbb{Z}} e_r\cdot w^{-r}\cdot w^t z^{-t}\cdot \frac{q^{2t}-q^{-2t}}{t} \\
&= e(w)\cdot \sum_{t>0} t^{-1}\left(\left(\frac{q^2w}{z}\right)^t - \left(\frac{q^{-2}w}{z}\right)^t\right) \\
&= e(w)\cdot F(z,w)
\end{aligned} \tag{42}$$

Note that

$$\begin{aligned}
e^{H(z)}\cdot e(w)\cdot e^{-H(z)} &= \left(1+H(z)+\frac{H^2(z)}{2!}+\cdots\right)e(w)\left(1-H(z)+\frac{H^2(z)}{2!}+\cdots\right) \\
&= e(w) + (H(z)e(w)-e(w)H(z)) \\
&\quad + (2!)^{-1}(H^2(z)e(w)+e(w)H^2(z)-2H(z)e(w)H(z)) + \cdots
\end{aligned} \tag{43}$$

In (43), note that the second term equals $e(w)\cdot F(z,w)$, and the third term equals

$$\begin{aligned}
&(2!)^{-1}\Big(H(z)\cdot[H(z),e(w)] - [H(z),e(w)]\cdot H(z)\Big) \\
=&\frac{F(z,w)}{2!}(H(z)e(w)-e(w)H(z)) = e(w)\cdot\frac{F^2(z,w)}{2!}.
\end{aligned} \tag{44}$$

We claim that the right-hand side of (43) equals $e(w)\cdot\sum_{t\geq 0}\frac{F^t(z,w)}{t!}$. Indeed, it is easy to check that the term with coefficient $\frac{1}{k!}$ in (43) is equal to $\frac{1}{k!}[H(z), e(w)F^{k-1}(z,w)]$, where the term $F^{k-1}(z,w)$ can be replaced by the terms with coefficient $\frac{1}{(k-1)!}$. □

Let us now show how such elements $H_{i,t}$ satisfying Lemma 25 can be obtained from the Cartan currents $\psi_i^{\pm}$.

Definition 26. *Define*

$$\overline{\psi}_i^{\pm}(z) = \frac{\psi_i^{\pm}(z)}{\psi_{i,0}^{\pm}} = 1 + \frac{\psi_{i,1}^{\pm}}{\psi_{i,0}^{\pm}}z^{\mp 1} + \frac{\psi_{i,2}^{\pm}}{\psi_{i,0}^{\pm}}z^{\mp 2} + \cdots$$

and define elements $\{H_{i,k}\}_{k\neq 0}$ via

$$\sum_{k\geq 1} H_{i,\pm k}z^{\mp k} = \pm log(\overline{\psi}_i^{\pm}(z))$$

Note that the quadratic relation (15) gives us

$$\psi_i^{\pm}(z)e_i(w) = e_i(w)\psi_i^{\pm}(z)\cdot\frac{q^2z-w}{z-q^2w} \tag{45}$$

Using (24), we obtain

$$\begin{aligned}\overline{\psi}_i^+(z)e_i(w) &= e_i(w)\overline{\psi}_i^+(z)\cdot\frac{1-q^{-2}wz^{-1}}{1-q^2wz^{-1}}\\ \overline{\psi}_i^-(z)e_i(w) &= e_i(w)\overline{\psi}_i^-(z)\cdot\frac{1-q^2w^{-1}z}{1-q^{-2}w^{-1}z}\end{aligned}\tag{46}$$

This indeed shows that $\exp H(z)$ and $\overline{\psi}_i^{\pm}(z)$ commute in the same fashion with $e_i(z)$. Therefore, when defining $H_{i,k}$ via $\pm log(\overline{\psi}_i^{\pm}(z))$ as in Definition 26, they should exactly satisfy the same commutation relations as the $H-$ terms in (40), not only for $k > 0$ but also for $k < 0$.

Similarly, one can show that for $i \neq j$ and any integers $s, t \in \mathbb{Z}$ for $t \neq 0$, we have:

$$[H_{i,t}, e_{j,s}] = \left(\frac{q_1^{-t}+q_3^{-t}-q_1^t-q_3^t}{t}\right)e_{j,s+t}\tag{47}$$

Now, with the relations between H-terms and e-terms figured out, it is time to unravel the h-terms. We wish to find h-terms obeying (34), and we will represent them using linear combination of the H-terms. This can be easily achieved assuming the matrix

$$\begin{bmatrix} q^{2t}-q^{-2t} & q_1^{-t}+q_3^{-t}-q_1^t-q_3^t\\ q_1^{-t}+q_3^{-t}-q_1^t-q_3^t & q^{2t}-q^{-2t}\end{bmatrix}\tag{48}$$

is non-degenerate for general choice of q, d ($q_3 = (dq)^{-1}, q_1 = dq^{-1}$), namely, set:

$$\begin{bmatrix} h_{0,t}\\ h_{1,t}\end{bmatrix} = t\cdot\begin{bmatrix} q^{2t}-q^{-2t} & q_1^{-t}+q_3^{-t}-q_1^t-q_3^t\\ q_1^{-t}+q_3^{-t}-q_1^t-q_3^t & q^{2t}-q^{-2t}\end{bmatrix}^{-1}\cdot\begin{bmatrix} H_{0,t}\\ H_{1,t}\end{bmatrix}\tag{49}$$

But for the determinant of (48) to vanish, d has to satisfy $(d^t + d^{-t})^2 = (q^t - q^{-t})^2$, hence it is generically non-zero. This completes our proof of Proposition 20.

3.3 Master Equality for $\mathfrak{gl}_2$

Similar to the $\mathfrak{gl}_1$ case treated in (Cui, 2021), the key vanishing property of the correlation functions for $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ crucially relies on the following "Master Equality":

Theorem 27 (Master Equality)**.**

$$\begin{aligned}\sum_{sym\{z_1,z_2\}}\frac{z_1-z_2}{z_1-q^2z_2}\times\Bigg(&\frac{(1+q^{-2})z_1-(q_1+q_3)w}{(z_1-q_1w)(z_1-q_3w)(z_2-q_1w)(z_2-q_3w)}\\ &+\frac{(1+q^2)z_1-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2}{(z_1-q_1w)(z_1-q_3w)(w-q_1z_2)(w-q_3z_2)}\\ &+\frac{(1+q^2)z_2-(q_1^{-1}+q_3^{-1})w}{(w-q_1z_1)(w-q_3z_1)(w-q_1z_2)(w-q_3z_2)}\Bigg)\\ =\sum_{sym\{z_1,z_2\}}\left(\frac{\alpha}{w}\cdot\delta(z_2,q^2z_1)\cdot\delta(z_1,q_3w)+\frac{\beta}{z_1}\cdot\delta(z_2,q_1w)\cdot\delta(w,q_3z_1)\right)&\end{aligned}\tag{50}$$

where

$$\alpha = \frac{1}{q_3-q_1},\qquad \beta = \frac{2+2q^{-2}-q_1-q_3-q_1^{-1}-q_3^{-1}}{q_2(1+q_2^{-1})(q_3-q_1)(q_3^2-q_2^{-1})}\tag{51}$$

Proof. Using the arguments of analytic continuation, it suffices to prove (50) under the condition that $|q_1|, |q_2| < 1$ and thus $|q_3| > 1$. Then, we switch any term of the denominator that cannot be represented by the same-named rational function in an open neighborhood of $z_1 = z_2 = w$ using delta-functions. Explicitly, if $|\gamma| > 1$, then we would replace $\frac{1}{a-\gamma b}$ by $\delta(a, \gamma b) - \frac{1}{\gamma b-a}$, see (9).

Lemma 28 (terms without delta-factors cancel)**.** *The following equality of rational functions holds:*

$$\sum_{sym\{z_1,z_2\}} \frac{z_1-z_2}{z_1-q^2z_2} \times \left(\frac{(1+q^{-2})z_1-(q_1+q_3)w}{(z_1-q_1w)(q_3w-z_1)(z_2-q_1w)(q_3w-z_2)} \right.$$
$$+\frac{(1+q^2)z_1-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2}{(z_1-q_1w)(q_3w-z_1)(w-q_1z_2)(q_3z_2-w)} \tag{52}$$
$$\left. +\frac{(1+q^2)z_2-(q_1^{-1}+q_3^{-1})w}{(w-q_1z_1)(q_3z_1-w)(w-q_1z_2)(q_3z_2-w)} \right) = 0$$

Proof. This result can be easily verified with Matlab. □

Lemma 29 (terms with two delta-factors cancel)**.** *The following equality holds:*

$$\sum_{sym\{z_1,z_2\}} \frac{z_1-z_2}{z_1-q^2z_2} \times \left(\frac{(1+q^{-2})z_1-(q_1+q_3)w}{(z_1-q_1w)(z_2-q_1w)} \cdot \delta(z_1,q_3w)\delta(z_2,q_3w) \right.$$
$$+\frac{(1+q^2)z_1-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2}{(z_1-q_1w)(w-q_1z_2)} \cdot \delta(z_1,q_3w)\delta(w,q_3z_2) \tag{53}$$
$$\left. +\frac{(1+q^2)z_2-(q_1^{-1}+q_3^{-1})w}{(w-q_1z_1)(w-q_1z_2)} \cdot \delta(w,q_3z_1)\delta(w,q_3z_2) \right) = 0$$

Proof. Actually, we claim that each summand in the LHS of (29) vanishes. For example, let us verify this for the first summand. We can replace z_1, z_2 both by q_3w because of the delta-factors, and hence we get 0 due to the factor $(z_1 - z_2)$. This idea can be similarly applied to the other two summands, where we replace two of the variables by the third and get a zero factor in front. □

Finally, let us consider the terms with a single delta-function factor: $\delta(z_i, q_3w)$ or $\delta(w, q_3z_i)$ with $i = 1$ or 2. For demonstration, let us look at the coefficient of $\delta(z_1, q_3w)$. It is precisely

$$-\frac{(z_1-z_2)((1+q^{-2})z_1-(q_1+q_3)w)}{(z_1-q^2z_2)(z_1-q_1w)(z_2-q_1w)(q_3w-z_2)}-$$
$$-\frac{(z_2-z_1)((1+q^{-2})z_2-(q_1+q_3)w)}{(z_2-q^2z_1)(z_2-q_1w)(z_1-q_1w)(q_3w-z_2)}+ \tag{54}$$
$$+\frac{(z_1-z_2)((1+q^2)z_1-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2)}{(z_1-q^2z_2)(z_1-q_1w)(w-q_1z_2)(q_3z_2-w)}$$

First, let us first replace z_1 by q_3w in (54) due to the delta-factor $\delta(z_1, q_3w)$, so that (54) is replaced by

$$-\frac{(q_3w-z_2)((1+q^{-2})q_3w-(q_1+q_3)w)}{(q_3w-q^2z_2)(q_3w-q_1w)(z_2-q_1w)(q_3w-z_2)}-$$
$$-\frac{(z_2-q_3w)((1+q^{-2})z_2-(q_1+q_3)w)}{(z_2-q^2q_3w)(z_2-q_1w)(q_3w-q_1w)(q_3w-z_2)}+ \tag{55}$$
$$+\frac{(q_3w-z_2)((1+q^2)q_3w-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2)}{(q_3w-q^2z_2)(q_3w-q_1w)(w-q_1z_2)(q_3z_2-w)}$$

Note that the only "bad" factor of (55) in an open neighborhood of $z_2 = w$ is $\frac{1}{z_2-q^2q_3w}$, as $|q^2q_3| = |q_1^{-1}| > 1$. Therefore, we have to replace it by $\delta(z_2, q_1^{-1}w) - \frac{1}{q_1^{-1}w-z_2}$. After this replacement, note that the terms without delta-factors vanish once again, i.e, we have the following equality of rational functions:

$$-\frac{(q_3w-z_2)((1+q^{-2})q_3w-(q_1+q_3)w)}{(q_3w-q^2z_2)(q_3w-q_1w)(z_2-q_1w)(q_3w-z_2)}-$$
$$-\frac{(z_2-q_3w)((1+q^{-2})z_2-(q_1+q_3)w)}{(q^2q_3w-z_2)(z_2-q_1w)(q_3w-q_1w)(q_3w-z_2)}+ \tag{56}$$
$$+\frac{(q_3w-z_2)((1+q^2)q_3w-(q_1+q_3+q_1^{-1}+q_3^{-1})w+(1+q^{-2})z_2)}{(q_3w-q^2z_2)(q_3w-q_1w)(w-q_1z_2)(q_3z_2-w)}=0.$$

On the other hand, after replacing z_2 by $q_1^{-1}w$ due to the delta-factor $\delta(z_2, q_1^{-1}w)$, (55) turns into

$$\frac{(q_1^{-1}w - q_3w)((1+q^{-2})q_1^{-1}w - (q_1+q_3)w)}{(q_1^{-1}w - q_1w)(q_3w - q_1w)(q_3w - q_1^{-1}w)} \cdot \delta(z_2, q_1^{-1}w) \tag{57}$$

which is equal to

$$\frac{1}{q_3 - q_1} \cdot \frac{1}{w} \cdot \delta(z_2, q_1^{-1}w). \tag{58}$$

It remains to note that $\delta(z_1, q_3w)\delta(z_2, q_1^{-1}w) = \delta(z_1, q_3w)\delta(z_2, q^2z_1)$.

Similarly, one can show that the coefficient of the delta-function term $\delta(w, q_3z_1)$ is equal to

$$\frac{2 + 2q^{-2} - q_1 - q_3 - q_1^{-1} - q_3^{-1}}{q_2(1+q_2^{-1})(q_3 - q_1)(q_3^2 - q_2)} \cdot \frac{1}{z_1} \cdot \delta(z_2, q_2^{-1}z_1) \tag{59}$$

This completes the proof of Theorem 27. □

3.4 Correlation Function of $\mathfrak{gl}_2$

Definition 30. *Let V be a $\mathbb{Z}$-graded representation of $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ with degrees bounded from above. For any $v \in V, \epsilon \in V^*$, consider the correlation function of the algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ defined via*

$$f\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right) = \left\langle \epsilon, e_{i_1}\left(z_1^{(i_1)}\right) \cdots e_{i_N}\left(z_N^{(i_N)}\right) v \right\rangle, \tag{60}$$

where $i_k \in \mathbb{Z}_2$ is the "color" of the formal variable z_k.

Again, the simple properties of all the correlation functions are similar to $n = 1$ and $n > 2$ cases:

Theorem 31.

$$f\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right) = \frac{\prod_{a<b}^{i_a=i_b}\left(z_a^{(i_a)} - z_b^{(i_b)}\right) \times A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)}{\prod_{a<b} g_{i_a i_b}\left(z_a^{(i_a)}, z_b^{(i_b)}\right)}. \tag{61}$$

where

$$g_{ii}(z, w) = z - q^2w, \qquad g_{ii+1}(z, w) = (z - dq^{-1}w)(z - (dq)^{-1}w), \tag{62}$$

and $A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)$ is a color-symmetric Laurent polynomial.

The proof of Theorem 31 is almost identical to that of Lemma 3.5 in (Cui, 2021) for the case of $\mathfrak{gl}_1$ and to the case of $\mathfrak{gl}_n (n > 2)$ as treated in the previous section. In addition, the Laurent polynomial A is subject to the key vanishing condition specified below:

Theorem 32. *If there exist 3 distinct variables $\left\{z_p^{(i_p)}, z_q^{(i_q)}, z_r^{(i_r)}\right\}$ where $i_p = i_q, i_r \neq i_p$, such that one of the following two conditions are met:*

1. $z_p^{(i_p)} = t, z_q^{(i_q)} = q^{-2}t, z_r^{(i_r)} = dq^{-1}t$
2. $z_p^{(i_p)} = t, z_q^{(i_q)} = q^{-2}t, z_r^{(i_r)} = d^{-1}q^{-1}t$

then the Laurent polynomial $A\left(z_1^{(i_1)}, \cdots, z_N^{(i_N)}\right)$ vanishes.

Proof. For simplicity, we shall consider the case of 3 variables involved (while the general case is treated completely analogously). We shall call $z_1^{(i_1)}, z_2^{(i_2)}, z_3^{(i_3)}$ simply by z_1, z_2, w.

Due to the General Cubic Serre relation (39), we know that the natural pairing between ϵ and ωv equals 0 where ω is the LHS of (39). Specifically,

$$\begin{aligned} A(z_1, z_2, w) \cdot \sum_{sym\{z_1,z_2\}} \frac{z_1 - z_2}{z_1 - q^2z_2} \times \Bigg(& \frac{(1+q^{-2})z_1 - (q_1+q_3)w}{(z_1 - q_1w)(z_1 - q_3w)(z_2 - q_1w)(z_2 - q_3w)} \\ & + \frac{(1+q^2)z_1 - (q_1 + q_3 + q_1^{-1} + q_3^{-1})w + (1+q^{-2})z_2}{(z_1 - q_1w)(z_1 - q_3w)(w - q_1z_2)(w - q_3z_2)} \\ & + \frac{(1+q^2)z_2 - (q_1^{-1} + q_3^{-1})w}{(w - q_1z_1)(w - q_3z_1)(w - q_1z_2)(w - q_3z_2)} \Bigg) = 0 \end{aligned} \tag{63}$$

Therefore, by the Master Equality (50), we get:

$$A\,(z_1, z_2, w) \sum_{sym\{z_1,z_2\}} \left(\frac{\alpha}{w} \cdot \delta(z_2, q^2 z_1) \cdot \delta(z_1, q_3 w) + \frac{\beta}{z_2} \cdot \delta(z_2, q_1 w) \cdot \delta(w, q_3 z_1) \right) = 0 \tag{64}$$

Thus, the vanishing condition of A follows from the linear independence of the double products of delta-factors, which is proven exactly as in Theorem 5.3 of (Cui, 2021) □

Remark. *At this point, it becomes clear that the Master Equality* (50) *did not emerge from nothing. It is essentially just the coefficient that we obtain after we establish the result for the natural pairing between the chosen covector and the vector with the LHS of the novel cubic Serre relation* (39)

Acknowledgement

I am particularly grateful to Professor Alexander Tsymbaliuk for his patient guidance and sincere advice. Also, I want to thank my parents for their support.

References

Cui, H. (2021). Correlation Functions of Quantum Toroidal $\mathfrak{gl}_1$ Algebra. *Journal of Mathematics Research, 13*(2), 1-7. https://doi.org/10.5539/jmr.v13n2p7

Enriquez, B. (2000). On correlation functions of Drinfeld currents and shuffle algebras. *Transformation Groups, 5*(2), 111-120. https://doi.org/10.1007/BF01236465

Feigin, B., Feigin, E., Jimbo, M., Miwa, T., & Mukhin, E. (2011). Quantum continuous gl: Semiinfinite construction of representations. *Kyoto Journal of Mathematics, 51*(2), 337-364. https://doi.org/10.1215/21562261-1214375

Feigin, B., Hashizume, K., Hoshino, A., Shiraishi, J., & Yanagida, S. (2009). A commutative algebra on degenerate CP 1 and Macdonald polynomials. *Journal of Mathematical Physics, 50*(9), 095215.

Feigin, B., Jimbo, M., Miwa, T., & Mukhin, E. (2016). Branching rules for quantum toroidal gln. *Advances in Mathematics, 300*, 229-274.

Odesskii, A. V., & Feigin, B. L. (1997). Elliptic deformations of current algebras and their representations by difference operators. *FUNCTIONAL ANALYSIS AND ITS APPLICATIONS C/C OF FUNKTSIONAL'NYI ANALIZ I EGO PRILOZHENIIA, 31,* 193-203. https://doi.org/10.1007/BF02465787

Ginzburg, V., Kapranov, M., & Vasserot, É. (1995). *Langlands reciprocity for algebraic surfaces*. arXiv preprint q-alg/9502013. https://doi.org/10.4310/MRL.1995.v2.n2.a4

Neguţ, A. (2020). Quantum toroidal and shuffle algebras. *Advances in Mathematics, 372,* 107288. https://doi.org/10.1016/j.aim.2020.107288

Negut, A. (2014). The shuffle algebra revisited. *International Mathematics Research Notices, 2014*(22), 6242-6275. https://doi.org/10.1093/imrn/rnt156

Miki, K. (2007). A (q, γ)-analog of the $W_{1+\infty}$ algebra. *J. Math. Phys., 12,* Paper 123520, 35 pp. https://doi.org/10.1063/1.2823979

Tsymbaliuk, A. (2018). PBWD bases and shuffle algebra realizations for $U_v(L\mathfrak{sl}_n), U_{v_1,v_2}(L\mathfrak{sl}_n), U_v(L\mathfrak{sl}(m|n))$ and their integral forms. arXiv preprint arXiv:1808.09536.