Specialization homomorphisms among shuffle algebras

Chuhan Wang

Abstract

We connect the shuffle algebras depending on a parameter $n$ to each other via specialization maps. In particular, we get $n^{2^m - 1}$ of algebra homomorphisms from the imaginary subalgebra of the $n$-th shuffle algebra to the smallest one. We also verify their compatibility with the previously studied Bethe, Heisenberg, and slope 0 subalgebras. Our approach is “dual” to [FJMM] that used fused currents.

1 Introduction

1.1 Summary of results

Shuffle algebras were first introduced by Feigin and Odesskii to study elliptic quantum groups corresponding to elliptic curves, see [FO]. At the same time, the classical quantum groups arose as $q$-deformation of universal enveloping Lie algebras in the 1970s, and since then, they have had many applications in maths and physics, namely in fields such as string theory, quantum field theory, combinatorics, inverse scattering method, geometric representation theory, quantization, and deformation theory. In the standard numerous literature the quantum groups are explicitly defined by generators and a long list of defining relations. The three common realizations of the $q$-deformed loop Lie algebra include the Drinfel'd-Jimbo realization, the new Drinfeld/loop realization, and the so-called RTT realization (see [DF] ). At the same time, as became clear over the last decade, the trigonometric version of the Feigin-Odesskii shuffle algebras provide a powerful combinatorial approach to quantum affine/loop groups, which is completely free of the cumbersome collections of generators and defining relations. In this realization, the algebras are realized in the space of multi-symmetric rational functions subject to some constraints on numerators and denominators, and with a rather simple but non-trivial shuffle product.

In the present paper, we connect various shuffle algebras among themselves in view of the above discussion. This can be viewed as a dual version of the construction of [FJMM] that embeds quantum toroidal (a.k.a. double affine/loop) $\mathfrak{gl}_1$ into the imaginary-degree part of the quantum toroidal $\mathfrak{gl}_n$ via
"fused currents". Since the latter objects are subject to certain convergence issues and may be defined only on a suitable category of representations, our combinatorial construction simplifies and generalizes that of [FJMM] as well as allows to establish a good behavior with respect to previously studied various important subalgebras in question.

1.2 Outline of the paper

The structure of the paper is as follows. We recall the definitions of shuffle algebras and quantum affine algebras along with other key elements of our setup in section 2. In Section 3, we introduce the first two main results of the paper in Main and Second Theorems. Then, we detail the main construction of our homomorphism mapping from shuffle algebras one dimension to shuffle algebras of another dimension with examples. This is where we will perform the main calculations that will give us the new parameters for the algebra. In Section 4, we verify the compatibility of our algebra homomorphisms with some of the known subalgebras, such as Bethe algebras and horizontal Heisenberg algebras. In Section 5, we outline some questions for future pursuits in this topic.

2 Set Up

2.1 Lie Algebras and Simple Lie Algebras

A Lie algebra \( \mathfrak{g} \) is a vector space over an arbitrary field \( \mathbb{F} \) with an operation:

\[
[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}
\]

called a Lie bracket, such that the following three properties hold:

1. This operation is bilinear: for \( \forall x, y, z \in \mathfrak{g} \), \( \forall a, b \in \mathbb{F} \)

\[
[ax + by, z] = a[x, z] + b[y, z] \\
[x, ay + bz] = a[x, y] + b[x, z]
\]

2. Skew-symmetry: \( [x, y] = -[y, x] \) \( \forall x, y \in \mathfrak{g} \)

3. Jacobi identity: \( [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \) \( \forall x, y, z \in \mathfrak{g} \)

A Lie algebra \( \mathfrak{g} \) is finite-dimensional simple if \( \mathfrak{g} \) is not abelian and the only ideals in \( \mathfrak{g} \) are 0 and \( \mathfrak{g} \) itself. The study of simple Lie algebras was carried out in the last century. In particular, using the root decomposition, one can classify simple Lie algebras by the combinatorial data of Dynkin diagrams or equivalently Cartan matrices. The latter can be used to present simple \( \mathfrak{g} \) by generators and relations.
2.2 Dynkin Diagram and Cartan Matrix

2.2.1 Dynkin Diagram

All simple Lie algebras are in bijection with one of such Dynkin diagrams and they are a way to encode roots of Lie algebras using nodes, therefore defining Lie algebras and providing insights into the representations of Lie groups. Each node in the diagram corresponds to a copy of \( \mathfrak{sl}_2 \). Two vertices are disconnected if the corresponding \( \mathfrak{sl}_2 \) commute and are joined by a single line whenever there is a nontrivial commutation relation. In this paper, we use \( I \) to represent the set of vertices of the Dynkin diagram of \( \mathfrak{g} \). The Dynkin diagrams with only one edge are known as simply laced Dynkin diagrams.

\[
A_{n-1}:egin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

\[
D_{n}:egin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

\[
E_{6}:egin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

Figure 1: Dynkin diagrams for some simply-connected simple Lie algebras

Here, the other Dynkin Diagrams of types \( B, C, D, E, G \) are left out of the picture because they are not simply laced. Dynkin diagrams are especially simple when they are representing simply laced Lie algebras. They are simple to treat because they don’t contain multiple edges, and the set of Serre relations are of either degree 2 or 3. In general setup, they can be much more complicated and the sub family of simply connected Lie algebras requires more a refined notion and all the combinatorial data about that.

A more specific way to construct Dynkin diagrams will be discussed below when we introduce Cartan Matrices.

2.2.2 Cartan matrix

In this paper, Cartan matrix will be denoted by \( A = (a_{ij})_{1,1}^{n} \) with \( n = |I| \) which is the number of vertices of the Dynkin diagram. \( A \) should be an symmetric \( n \times n \) positive definite square matrix subject to the following relations:

\[
\begin{cases}
  a_{ii} = 2 \\
  a_{ij} \in \mathbb{Z}_{\leq 0} & \forall i \neq j \\
  a_{ij} = 0 \Leftrightarrow a_{ji} = 0
\end{cases}
\]

We note that Cartan matrices of simple Lie algebras are diagonalizable, i.e., there exist relatively prime positive integers \( d_i \) such that \( d_i a_{ij} = d_j a_{ji} \). The symmetrized Cartan matrix \( (d_{ij}) \) is defined to have entries given by these values \( d_i a_{ij} \). A crucial property of the Cartan matrices is that it defines a positive definite quadratic form. That is, we have the inequality \( \sum_{i=1}^{n} d_{ij} x_i x_j \geq 0 \) for any \( x_1, \ldots, x_n \in \mathbb{R} \) with an equality only for \( x_1 = \ldots = x_n = 0 \). For Cartan
matrices of simply laced Dynkin Diagrams, all \( d_i = 1 \). For that reason, there’s no real difference for \( d_{ij} \) and \( a_{ij} \) in simply laced cases. Replacing the condition of positive definite by positive semidefinite forms (where the equality holds not only for the zero vector) gives rise to affine/loop algebras that will be introduced below in a different realization.

### 2.3 Tensor Algebra

Let \( V \otimes k \) denote the \( k \)-fold tensor product of the vector space \( V \), defined over a field \( F \). The tensor algebra \( T(V) \) is defined to be the vector space

\[
T(V) = F \oplus \bigoplus_{n=1}^{\infty} V \otimes n
\]

with sum between \( V \otimes k \) as direct sums and multiplication defined by external tensor product, namely

\[
(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_m) = a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m.
\]

Once a basis \( \{e_1 \cdots e_n\} \) of \( V \) is chosen, \( T(V) \) is naturally identified with the free associative algebra generated by \( \{e_1 \cdots e_n\} \) with a multiplication being a concatenation of words.

If \( g \) is a Lie algebra with a basis of \( \{e_1 \cdots e_n\} \), the universal enveloping algebra \( U(g) \) is defined as the quotient of \( T(g) \) by two-sided ideal generated by

\[
e_i \otimes e_j - e_j \otimes e_i - [e_i, e_j].
\]

This construction is actually independent of a choice of a basis, as follows from the universal property of \( U(g) \):

\[
\text{Hom}_{\text{Lie algebra}}(g, A) = \text{Hom}_{\text{Associative algebra}}(U(g), A)
\]

for any associative algebra \( A \), endowed with a Lie bracket via \([a, b] = ab - ba\).

### 2.4 Loop Lie Algebras

For any Lie algebra \( g \), the loop Lie algebra \( Lg \) is defined as the vector space:

\[
Lg = g \otimes \mathbb{C}[t, t^{-1}] = \left\{ a(t) = \sum_n a_n t^n \big| a_n \in g, n \in \mathbb{Z}, a_n = 0 \text{ for } |n| \gg 0 \right\}
\]

endowed with the Lie bracket:

\[
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} \quad \forall x, y \in g, \forall n, m \in \mathbb{Z}
\]

For example, if \( g = \mathfrak{sl}_2 \), the traceless \( 2 \times 2 \) matrices which have the basis

\[
e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

then the loop algebra \( L(\mathfrak{sl}_2) \) has a basis \( \{et^n, ft^n, ht^n\}_{n \in \mathbb{Z}} \).

### 2.5 Drinfeld-Jimbo Quantum Group

We define the \( q \)-integers, factorials, and binomial coefficients via:

\[
[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]!_q = [1]_q \cdots [k]_q, \quad \left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{[n]_q \cdots [n-k]_q}{[k]_q \cdots [n-k]_q}
\]
For a finite-dimensional simple Lie algebra \( \mathfrak{g} \), the Drinfeld-Jimbo quantum group \( U_q(\mathfrak{g}) \) is generated by \( \{ E^\pm_i, K^\pm_i \}_{i \in I} \), with \( I \) denoting the set of the vertices of the Dynkin diagram of \( \mathfrak{g} \) as before and the following defining relations (for all \( i, j \in I \)).

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{1-a_{ij}}{k} \right) (E^+_i)^k (E^-_i)^{1-a_{ij}-k} = 0, \quad i \neq j \tag{1}
\]

\[
K_j E^+_i = q^{a_{ij}} E^+_i K_j, \quad K_i K_j = K_j K_i \tag{2}
\]

\[
[E^+_i, E^-_j] = \delta^j_i \left( \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \right) \tag{3}
\]

Here, as agreed above, \( (a_{ij}) \) denotes the entry of Cartan matrix of \( \mathfrak{g} \) and \( (d_{ij}) \) denotes that of the symmetrized Cartan matrix of \( \mathfrak{g} \). The relations (1) are commonly called \( q \)-Serre relations.

### 2.6 Quantum Loop Algebra

The quantum loop algebra \( U_q(\mathcal{L}\mathfrak{g}) \) is a quantization of the universal enveloping algebra of \( \mathcal{L}\mathfrak{g} \). We can define it by generators

\[
\{ E^\pm_{i,r}, K^\pm_{i,r}, q^{\pm e} \}_{r \in \mathbb{Z}, i \in I} \]

satisfying the following relations. The relations are:

\[
e^+_i(z)e^+_j(w) (w - zq^{-d_{ij}}) = e^+_j(w)e^+_i(z) (wq^{-d_{ij}} - z) \tag{4}
\]

\[
e^-_i(z)e^-_j(w) (w - zq^{d_{ij}}) = e^-_j(w)e^-_i(z) (wq^{d_{ij}} - z) \tag{5}
\]

if \( i \neq j \)

\[
\sum_{\sigma \in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{1-a_{ij}}{k} \right) e^+_i \left( z_{\sigma(1)} \right) \ldots e^+_i \left( z_{\sigma(k)} \right) e^+_j \left( w \right) e^+_i \left( z_{\sigma(k+1)} \right) \ldots e^+_i \left( z_{\sigma(1-a_{ij})} \right) = 0 \tag{6}
\]

if \( i \neq j \)

\[
\sum_{\sigma \in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{1-a_{ij}}{k} \right) e^-_i \left( z_{\sigma(1)} \right) \ldots e^-_i \left( z_{\sigma(k)} \right) e^-_j \left( w \right) e^-_i \left( z_{\sigma(k+1)} \right) \ldots e^-_i \left( z_{\sigma(1-a_{ij})} \right) = 0 \tag{7}
\]

\[
K^+_j(z)K^+_i(z) (z - wq^{-d_{ij}}) = e^+_j(z)K^+_i(z) (zq^{-d_{ij}} - w) \tag{8}
\]

\[
K^+_j(z)K^-_i(z) (z - wq^{d_{ij}}) = e^-_j(z)K^-_i(z) (zq^{d_{ij}} - w) \tag{9}
\]

\[
K^+_i(z)K^+_j(z) = K^+_j(z)K^+_i(z), \quad K^+_{i,0}K^-_{i,0} = 1 \tag{10}
\]

\[
[e^+_i(z), e^-_j(w)] = \frac{\delta^j_i \left( \frac{z}{w} \right)}{q_i - q_i^{-1}} \cdot (K^+_i(z) - K^-_i(w)) \tag{11}
\]

where \( q_i = q^{d_{ij}} \). We first define the currents as follows:

\[
e^+_i(z) = \sum_{r \in \mathbb{Z}} E^+_i z^{-r}, \quad e^-_i(z) = \sum_{r \in \mathbb{Z}} E^-_i z^{-r}, \quad K^\pm_i(z) = \sum_{l \in \mathbb{N}} K^\pm_{i,l} z^{-l}
\]
We note that the assignment $E^+_i \mapsto E^+_{i,0}$, $E^-_i \mapsto E^-_{i,0}$ and $K^\pm_{i,j} \mapsto K^\pm_{i,j,0}$ gives rise to a injective algebra homomorphism: $U_q(\mathfrak{g}) \to U_q(\mathfrak{L})$. Evoking the triangular decomposition:

$$U_q(\mathfrak{L}) \cong U_q(\mathfrak{L})^+ \otimes U_q(\mathfrak{L})^0 \otimes U_q(\mathfrak{L})^- \quad (12)$$

where the middle factor is commutative, while the third factor is naturally isomorphic to the first one via $E^-_{i,r} \mapsto E^-_{i,-r}$, we shall just focus on the positive subalgebra $U_q(\mathfrak{L})^+$ generated by $E^+_{i,j}$ with $i \in I$, $r \in \mathbb{Z}$.

### 2.7 Quantum Toroidal Algebras of $\mathfrak{gl}_n$

The crucial feature of the loop algebras $\mathfrak{L}$ for simple $\mathfrak{g}$ is that they can also be defined by generators and relations, similarly to $\mathfrak{g}$, but starting from the extended Dynkin diagram or extended Cartan matrix corresponding to positive semidefinite quadratic forms, see 2.2. As such taking the loops in them, one obtains the toroidal algebras $U_q(\widehat{\mathfrak{g}})$. However, only in type $A_n$, it is also possible to introduce yet another deformation parameter, which corresponds to having cycles in the corresponding extended Dynkin diagram. This results in the quantum toroidal algebras $U_{q,d}(\widehat{\mathfrak{gl}}_n)$ that underline the present work. In the rest of the paper we shall however use their combinatorial realizations, evoked in the Introduction, which we recall in the next section.

### 2.8 Shuffle algebras

Shuffle algebras were first introduced by Feigin and Odesskii to study elliptic quantum groups corresponding to elliptic curves, see [FO]. Let $\Sigma_k$ denote the symmetric group of $k$ elements, and $\Sigma(k_1, \ldots, k_n) = \Sigma_{k_1} \times \cdots \times \Sigma_{k_n}$ for $k_1, \ldots, k_n \in \mathbb{N}$. Consider an $\mathbb{N}$-graded vector space $S^{(n)} = \bigoplus \mathbb{S}^{(n)}(k)$, where $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $\mathbb{S}^{(n)}(k)$ consists of $k$ symmetric rational functions in the variables $\{x_{i,r}\}_{i \in I}$ for $1 \leq r \leq k_i$. Here, $n = |I|$ as before. In this paper, we use $X_{i,r}^{i'k_i}$ to represent $\{x_{i,r}\}_{i \in I}$. For any collection of rational functions $\zeta_{ij}(z)$, we endow the shuffle algebra $S^{(n)}$ with the following bilinear shuffle product $*$: given $F \in \mathbb{S}^{(n)}_k$ and $G \in \mathbb{S}^{(n)}_l$, define $F * G \in \mathbb{S}^{(n)}_{k+l}$

$$(F * G) \left( X_{1,1}^{i_1 k_1 + l_1}, \ldots, X_{n-1,1}^{i_{n-1} k_{n-1} + l_{n-1}} \right) = \frac{1}{k_1! \cdot l_1!} \cdot \text{Sym}_{\Sigma_{k+l}} \left( F \left( X_{i,1}^{i,k_i} \right) \cdot G \left( X_{j,1}^{j,k_j + l_j} \right) \cdot \prod_{i \in I} \prod_{r \leq k_i} \zeta_{i,j} \left( \frac{x_{i,r}}{x_{j,s}} \right) \right)$$

where $k_1! = \prod_{i \in I} k_i!$, and the symmetrization of $F \in \mathbb{C} \left( X_{i,1}^{i,k_i} \right)$ is defined via

$$\text{Sym}_{\Sigma_k} (F) \left( X_{i,1}^{i,k_i} \right) = \sum_{(\sigma_1, \ldots, \sigma_{k-1}) \in \Sigma_k} F \left( X_{i,\sigma_i(k_i)} \right)$$

, the averaging of $F$ over the product of symmetric groups acting termwise by permutation of the variables of each color.
In the following four sections 2.8.1–2.8.4, we introduce the shuffle factors as well as pole and wheel conditions for all $n$ with $n \geq 1$. While the treatments of $n=1$, $n=2$, $n=3$, and $n>3$ cases are different, they naturally fit into the same framework and have a common underlying picture and logic.

2.8.1 Shuffle Algebra for $\mathfrak{gl}_1$

If $g = \mathfrak{gl}_1$, since there is only one color, we let $k_1 = k$, $\{x_{1,r}\}_{r=1}^{k} = \{x_{r}\}_{r=1}^{k}$. If $f(X^1_1) = \prod_{1 \leq i \neq j \leq k} (x_i - x_j)$, and $F$ is a symmetric Laurent polynomial, then $f$ is said to satisfy the pole condition.

If $f(X^1_1) = 0$ whenever there exist $1 \leq a, b, c \leq k$ such that $x_a = qd^{-1}x_b$, $x_b = qdx_c$ and $x_c = q^{-2}x_a$, then $f$ is said to satisfy the wheel condition. These ratios are conveniently encoded by the weighted edges as depicted in Figure 2.

Let $Sh_{\mathfrak{gl}_1}^\approx(k)$ denote the subspace of all $f \in S$ satisfying both pole and wheel conditions.

![Figure 2: Wheel condition-\(Sh_{\mathfrak{gl}_1}^\approx(k)\)](image)

The shuffle factor of $Sh_{\mathfrak{gl}_1}^\approx(k)$ is given by

$$\omega^{(1)} \left( \frac{x_i}{x_j} \right) = \frac{(x_i - q^{-2}x_j)(x_i - qd^{-1}x_j)(x_i - qdx_j)}{(x_i - x_j)^3}$$

$$= \frac{(x_i - q^{-1}x_j)(x_i - q^{-1}q^{-1}x_j)(x_i - q^{-1}x_j)}{(x_i - x_j)^3}$$

2.8.2 Shuffle Algebra for $\mathfrak{gl}_2$

If $g = \mathfrak{gl}_2$, we let $\{x_{1,r}\}_{r=1}^{k_1} = X_{1,1}^{1,k_1}$ and $\{x_{2,s}\}_{s=1}^{k_2} = X_{2,1}^{2,k_2}$, specify $X_{1,1}^{1,k_1}$ color 0, and $X_{2,1}^{2,k_2}$ color 1.

If $f(X_{1,1}^{1,k_1} | X_{2,1}^{2,k_2}) = \prod_{1 \leq i \neq j} (x_{1,i} - x_{2,j})$, and $F$ is a symmetric Laurent polynomial, then $f$ is said to satisfy the pole condition.

If $f(X_{1,1}^{1,k_1} | X_{2,1}^{2,k_2}) = 0$ whenever there exist $1 \leq a, c \leq k_1$ and $1 \leq b \leq k_2$ meet one of following condition 1 or condition 2, then $f$ is said to satisfy the wheel condition and we say $f$ wheel in color $(0,0,1)$. 

7
If \( f \left( X_1^{k_1} X_2^{k_2} \right) = 0 \) whenever there exist \( 1 \leq b \leq k_1 \) and \( 1 \leq a, c \leq k_2 \) meet one of following condition 1 or condition 2, then \( f \) is said to satisfy the pole condition, and we say \( f \) wheel in color \((1, 1, 0)\).

\[
\begin{align*}
\text{condition 1: } & x_{1,a} = qdx_{2,b} \quad \text{and} \quad x_{2,b} = qdx_{1,c} \\
\text{condition 2: } & x_{1,a} = qd^{-1}x_{2,b} \quad \text{and} \quad x_{2,b} = qdx_{1,c}
\end{align*}
\]

These ratios are conveniently encoded by the weighted edges as depicted in Figure 3.

Figure 3: Wheel condition-\( \text{Sh}_{\mathfrak{gl}_2} \)

Let \( \text{Sh}_{\mathfrak{gl}_2} \) denote the subspace of all \( f \in \mathcal{S} \) satisfying both pole and wheel conditions. The shuffle factor of \( \text{Sh}_{\mathfrak{gl}_2} \) is given by

\[
\omega^{(2)}_{i,j} \left( \frac{x_{i,r}}{x_{j,s}} \right) = \begin{cases} 
\frac{x_{i,r} - q^{-2}x_{j,s}}{x_{i,r} - x_{j,s}}, & \text{if } j \equiv i \mod 2; \\
\frac{\left( x_{i,r} - qd^{-1}x_{j,s} \right) \left( x_{i,r} - qdx_{j,s} \right)}{(x_{i,r} - x_{j,s})^2}, & \text{if } j \equiv i \pm 1 \mod 2;
\end{cases}
\]

That is:

\[
\omega^{(2)}_{i,j} \left( \frac{x_{i,r}}{x_{j,s}} \right) = \begin{cases} 
\frac{x_{i,r} - q_{2}^{-1}x_{j,s}}{x_{i,r} - x_{j,s}}, & \text{if } j \equiv i \mod 2; \\
\frac{\left( x_{i,r} - q^{-1}x_{j,s} \right) \left( x_{i,r} - q_{2}^{-1}x_{j,s} \right)}{(x_{i,r} - x_{j,s})^2}, & \text{if } j \equiv i \pm 1 \mod 2;
\end{cases}
\]

2.8.3 Shuffle Algebra for \( \mathfrak{gl}_3 \)

If \( \mathfrak{g} = \mathfrak{gl}_3 \), we let \( \{ x_{1,i} \}_{i=1}^{k_1} = X_1^{k_1} \), \( \{ x_{2,i} \}_{i=1}^{k_2} = X_2^{k_2} \), and \( \{ x_{3,i} \}_{i=1}^{k_3} = X_3^{k_3} \). Specify \( X_1^{k_1} \) color 0, \( X_2^{k_2} \) color 1 and \( X_3^{k_3} \) color 2.

If \( f \left( X_1^{k_1} X_2^{k_2} X_3^{k_3} \right) = \prod_{1 \leq i \leq k_1} (x_{1,i} - x_{2,i}) \cdot \prod_{1 \leq i \leq k_2} (x_{2,i} - x_{3,i}) \cdot \prod_{1 \leq i \leq k_3} (x_{3,i} - x_{1,i}) \), and

\[ F \] is a symmetric Laurent polynomial, then \( f \) is said to satisfy the pole condition.
If \( f \left( X_{1,1}^{1,k_1} | X_{2,1}^{2,k_2} | X_{3,1}^{3,k_2} \right) = 0 \) whenever there exist \( 1 \leq a, c \leq k_i \) and \( 1 \leq b \leq k_j \) meet one of following condition 1 or condition 2, then \( f \) is said to satisfy the wheel condition and we say \( f \) wheel in color \((i-1,i-1,j-1)\), where \( 1 \leq i, j \leq 3 \).

condition 1: \( j \equiv i + 1 \mod 3 \), \( x_{i,a} = qdx_{j,b} \) and \( x_{j,b} = qd^{-1}x_{i,c} \)
condition 2: \( j \equiv i - 1 \mod 3 \), \( x_{i,a} = qd^{-1}x_{j,b} \) and \( x_{j,b} = qdx_{i,c} \)

Let \( Sh_{\mathfrak{gl}_n} (k_1, k_2, k_3) \) denote the subspace of all \( f \in S \) satisfying both pole and wheel conditions. The shuffle factor of \( Sh_{\mathfrak{gl}_n} \) is given by

\[
\omega_{1,j}^{[3]} \left( \frac{x_{i,r}}{x_{j,s}} \right) = \begin{cases} 
x_{i,r} - q^{-2} x_{j,s}, & \text{if } j \equiv i \mod 3; \\
x_{i,r} - x_{j,s}, & \text{if } j \equiv i + 1 \mod 3; \\
d^{-1} x_{i,r} - q x_{j,s}, & \text{if } j \equiv i - 1 \mod 3;
\end{cases}
\]

2.8.4 Shuffle Algebra for \( \mathfrak{gl}_n \)

If \( \mathfrak{g} = \mathfrak{gl}_n \), we let \( \{x_{1,r}\}_{r=1}^{k_1} = X_{1,1}^{1,k_1} \ldots, \{x_{n,r_n}\}_{r_n=1}^{k_n} = X_{n,1}^{n,k_n} \), and specify \( X_{1,1}^{1,k_1} \) color 0, \ldots, \( X_{n,1}^{n,k_n} \) color \( n-1 \).

If \( f \left( \prod_{1 \leq r < s \leq k_i} (x_{1,r} - x_{s,j}) \right) = 0 \) whenever there exist \( 1 \leq a, c \leq k_i \) and \( 1 \leq b \leq k_j \) meet one of following condition 1 or condition 2, then we say the shuffle algebra wheel in color \((i-1,i-1,j-1)\), where \( 1 \leq i, j \leq n \).

condition 1: \( j \equiv i + 1 \mod n \), \( x_{i,a} = qdx_{j,b} \) and \( x_{j,b} = qd^{-1}x_{i,c} \)
condition 2: \( j \equiv i - 1 \mod n \), \( x_{i,a} = qd^{-1}x_{j,b} \) and \( x_{j,b} = qdx_{i,c} \)

These ratios are conveniently encoded by the weighted edges as depicted in Figure 4. Let \( Sh_{\mathfrak{gl}_n} (k_1, \ldots, k_n) \) denote the subspace of all \( f \in S^k \) satisfying both pole and wheel conditions. For \( 1 \leq r \leq k_i \) and \( 1 \leq s \leq k_j \), the shuffle factor of \( Sh_{\mathfrak{gl}_n} \) is given by

\[
\omega_{1,j}^{[n]} \left( \frac{x_{i,r}}{x_{j,s}} \right) = \begin{cases} 
x_{i,r} - q^{-2} x_{j,s}, & \text{if } j \equiv i \mod n; \\
x_{i,r} - x_{j,s}, & \text{if } j \equiv i + 1 \mod n; \\
d^{-1} x_{i,r} - q x_{j,s}, & \text{if } j \equiv i - 1 \mod n; \\
x_{i,r} - qd^{-1} x_{j,s}, & \text{otherwise}.
\end{cases}
\]
The relation between the shuffle and quantum toroidal algebras is established by the following results.

**Proposition 2.9** The assignment $E_{i,r}^+ \mapsto x_{i,1}^r$ give rise to an algebra homomorphism $\rho: U_{q,d}(\hat{g}_t) \to \mathcal{Sh}_{\mathfrak{gl}_n}$

The proof of this result is straightforward as it requires only verifying that the above assignment is compatible with the defining quadratic and cubic relations. The following result is much harder and was proved in [N1, N2]

**Theorem 2.10** The homomorphism $\rho$ in the Proposition 2.9 is actually an algebra isomorphism.

### 3 Main Results

The paper of [FJMM] provided an interesting but quite obscure construction of the algebra homomorphisms from the quantum toroidal of $\mathfrak{gl}_1$ into the imaginary degree part of the quantum toroidal $\mathfrak{gl}_n$. The latter involved fusion of currents, which involves infinite sums, hence being defined only up to convergence assumptions.

We shall start our treatment from its simplest counterpart, namely the simplest non-trivial case, the map from $\bigoplus_k \mathcal{Sh}_{\mathfrak{gl}_n} \to \mathcal{Sh}_{\mathfrak{gl}_2}$. This simplest case contains many of the key elements of our general construction and provides insight into the homomorphism, and how different inputs lead to predictable numbers of specializations. One of the goals is to construct an algebra homomorphism from the imaginary subalgebra of a shuffle algebra to the whole of another, mathematically speaking it is: $\mathcal{Sh}_{\mathfrak{gl}_m} \to \mathcal{Sh}_{\mathfrak{gl}_n}$. For our main theorem, we generalize the discussion in [FHHSSY] and construct algebra homomorphisms from certain subalgebras of $\mathcal{Sh}_{\mathfrak{gl}_n}$ to $\mathcal{Sh}_{\mathfrak{gl}_m}$. There’s a family of algebra homomorphisms given by certain specialization procedure that would be described later:
Theorem 3.1 (Main Theorem)

We have algebra homomorphism:

\[ \Upsilon_{m+n,n}: \bigoplus_{k_0, k_1, \ldots, k_{n-1} \in \mathbb{N}} Sh_{\mathfrak{g}_{m+n}} \mathfrak{gl}_{m+n} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1}) |_{q_1, q_2, q_3} \to Sh_{\mathfrak{g}_{l_n}} |_{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3} \]

Theorem 3.2 (Second Theorem)

We have algebra homomorphism:

\[ \Upsilon_n : \bigoplus_{k \in \mathbb{N}} Sh_{\mathfrak{g}_{l_n}} (k, \ldots, k) |_{q_1, q_2, q_3} \to Sh_{\mathfrak{g}_{l_1}} |_{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3} \]

Remark 3.3 We note that the above two algebras both in Main Theorem and Second Theorem will have different parameters as the homomorphism doesn’t preserve the parameters and there are different parameters \( q_1, q_2, q_3 \) and \( \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \) for different mappings depending on specific homomorphism constructions. The mapping is component-wise.

We will verify in Section 4 that these algebra homomorphisms with some compatibility of subalgebras studied in [N1] [FT]. In Section 4, we shall also verify that these homomorphisms are compatible with important subalgebras studied in [FHHSY, FT, N1].

3.4 From \( Sh_{\mathfrak{g}_{l_2}} \) to \( Sh_{\mathfrak{g}_{l_1}} \)

By considering the typical of \( Sh_{\mathfrak{g}_{l_2}} \) to \( Sh_{\mathfrak{g}_{l_1}} \), we can see that our maps of specialization will be of the following nature of specializing a segment of the variables and then multiplying by some rational function. From now on, we shall always define some specialization of a function \( f \) as \( \text{spec} \cdot f \). In this case, we send \( f \) to its specialization times \( G \), \( f \mapsto \text{spec}(f) \cdot G \). Here, the specialization action is taking some of the variables to be multiples of each other and \( g \) as some rational function factor. See the later section for more details.

In \( Sh_{\mathfrak{g}_{l_2}} (k, k) \), we have variables of \( X_{1,1}^{1,k} \) and \( X_{2,1}^{2,k} \) in two different colors. To simplify notations, we rename \( X_{1,1}^{1,k} \) and \( X_{2,1}^{2,k} \) by \( X_{1}^{k} \) and \( Y_{1}^{k} \) respectively. Further, in this paper, we use \( A_{i}^{j} \) to represent \( (a_{i}, a_{i+1} \ldots a_{j}) \), where \( a_i \) is a variable, and use \( cA_{i}^{j} \) to represent \( (ca_{i}, ca_{i+1} \ldots ca_{j}) \), where \( c \) is a constant.

In order to construct an algebra homomorphism \( \Upsilon_{2,1} : Sh_{\mathfrak{g}_{l_2}} (k, k) \to Sh_{\mathfrak{g}_{l_1}} (k) \) the general idea of our algorithm is: for \( \forall f(X_{1}^{k}|Y_{1}^{k}) \in Sh_{\mathfrak{g}_{l_2}} (k, k) \) we specialize \( X_{1}^{k} = \lambda Y_{1}^{k} \), therefore, \( \Upsilon_{2,1}(f(X_{1}^{k}|Y_{1}^{k})) = f(\lambda Y_{1}^{k}|Y_{1}^{k}) \).

We know \( f(X_{1}^{k}|Y_{1}^{k}) \) satisfies the pole condition and wheel condition of \( Sh_{\mathfrak{g}_{l_2}} \), while \( f(\lambda Y_{1}^{k}|Y_{1}^{k}) \) satisfies those conditions of \( Sh_{\mathfrak{g}_{l_1}} \), we can amend this by using a rational factor of \( G(Y_{1}^{k}) \).
In addition, we also anticipate some constants that inherently come with the specializations, which we would denote $c_k$ for a function $f(X_1^k)$. Our goal is to determine the rational factor $G(Y_1^k)$, $c_k$, and explicit relations between $q_1, q_2, q_3$ and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ that would make $\Upsilon_{2,1}$ a compatible homomorphism mapping.

**Remark 3.5** Our main argument for why $\lambda \in \{qd^{\pm 1}, qd^{\pm 1}\}$ comes from analyzing the number of terms in both sides of the homomorphism. We have homomorphism, $\Upsilon(f(x) \ast g(y))$, for $f, g$ functions of $k, l$ variables respectively is $C_{k+l}^k \cdot C_{k+l}^l$; and such choices of $\lambda$ to kill the unbalanced terms. On the LHS we are picking a set of $k$ elements of a specific color to go into $f(x)$ from all elements of that color from $f(x)$ and $g(y)$, yielding $C_{k+1}^k$. On the RHS, $\Upsilon f(x) \ast \Upsilon g(x)$ only has two terms. Therefore, make the number of terms on both sides equal by letting the $\omega$ terms on the left vanish, restricting the possible choices of $\lambda$ to $qd, qd^{-1}$.

For example. For $f(x_1|y_1), g(x_2|y_2) \in Sh_{\mathbb{P}_2}^1(1, 1)$, let $x_i = \lambda y_i$ with $i = 1, 2$. To simplify the notations, we omit the shuffle factors and $G$ factors in computing $f \ast g$ by using the symbol $\sim$ instead of equality. To ensure $2, 1(f) = 2, 1(g)$, we must kill the terms containing $f(y_2|y_1) \cdot g(y_1|y_2)$ and $f(y_1|y_2) \cdot g(y_2|y_1)$. Tracing back the zeroes of the corresponding $\omega(2)$ factors we find that $\sim qd$ or $\sim qd^{-1}$. We want to construct a homomorphism $\Upsilon_{2,1}$ such that $\Upsilon_{2,1}(f \ast g) \sim \Upsilon_{2,1}(f) \ast \Upsilon_{2,1}(g)$.

$$\Upsilon_{2,1}(f \ast g) \sim f(\lambda y_1|y_1) \cdot g(\lambda y_2|y_2) + f(\lambda y_2|y_2) \cdot g(\lambda y_1|y_1) + f(\lambda y_2|y_1) \cdot g(\lambda y_1|y_2) + f(\lambda y_1|y_2) \cdot g(\lambda y_2|y_1)$$

$$\Upsilon_{2,1}(f) \ast \Upsilon_{2,1}(g) \sim f(y_1) \ast g(y_2) \sim f(y_1) \cdot g(y_2) + f(y_2) \cdot g(y_1)$$

To ensure $\Upsilon_{2,1}(f \ast g) \sim \Upsilon_{2,1}(f) \ast \Upsilon_{2,1}(g)$, we must kill the item of $f(\lambda y_2|y_1) \cdot g(\lambda y_2|y_2)$ and $f(\lambda y_1|y_2) \cdot g(\lambda y_2|y_1)$, which can be done by specializing $\lambda \in \{qd^{\pm 1}, qd^{\pm 1}\}$.

In addition, we can prove the validity of our $\lambda$ choices by looking at the wheel conditions. We specify $X_1^k$ color $0$ and $Y_1^k$ color $1$. To map two colors to one single color, we have to make sure that the image of every shuffle product between two variables of different colors should be contained in $Sh_{\mathbb{P}_1}(k)$. This means $\omega(2)(\frac{x}{y}) = 0$, so $(x - qd^{-1}y)(x - qdy) = 0$, implying that $x = qd^{-1}y$ or $x = qdy$ and $\lambda = qd$ or $\lambda = qd^{-1}$.

In summary, these $\lambda$ ensure that we kill unwanted wheel conditions that arose from different colors but also preserve the homomorphism of the mapping.

Below, we will discuss how choices of $\lambda$ yields different rational factor $G$ and different relation between the parameters $q_1, q_2, q_3$ and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$: 
3.5.1 Case 1: $\lambda = qd$

In this case, $X^k_\lambda = qdY^k_\lambda$. First, since $\Upsilon_{2,1}$ is an algebra homomorphism from $Sh_{\varnothing_2}$ to $Sh_{\varnothing_1}$, we must have $\prod_{1 \leq i \neq j \leq k} (y_i - y_j)$ in the denominator, and have $\prod_{1 \leq i,j \leq k} (qd y_i - y_j)^2$ in the numerator to eliminate the denominator of $f(qdY^k_\lambda | Y^k_\lambda)$. Second, the specialization of $f$ has certain zero divisors due to the wheel conditions of $Sh_{\varnothing_1}$. Explicitly, we have zeroes at any $y_i - q^2 y_j$ for $i \neq j$. Thus, we try

$$G(Y^k_\lambda) = \frac{\prod_{1 \leq i,j \leq k} (qd y_i - y_j)^2}{\prod_{1 \leq < j \leq k} (y_i - y_j)^2 (y_i - q^2 y_j)(y_i - q^{-2} y_j)} \cdot C_k$$

$C_k$ here is a constant that is yet to be determined, which depends on $q$ and $d$ but not $Y^k_\lambda$. In particular, when $k = 1$,

$$\Upsilon_{2,1}(f(x | y)) = \frac{p(qd y | y)}{(qd y - y)^2} \cdot (qd y - y)^2 \cdot C_1 = p(qd y | y) \cdot C_1$$

Given $f(X^k_\lambda | Y^k_\lambda) \in Sh_{\varnothing_2} (k, k)$ and $g(X^{k+l}_{k+1} | Y^{k+l}_{k+1}) \in Sh_{\varnothing_1} (l, l)$, we have

$$\Upsilon_{2,1}(f \ast g) = Sym \left[ \frac{f \left( qdY^k_\lambda | Y^k_\lambda \right) \cdot g \left( qdY^{k+l}_{k+1} | Y^{k+l}_{k+1} \right) \cdot \omega^{(2)}_{i,j} \left( \frac{qdY^k_\lambda}{qdY^{k+l}_{k+1}} \right)}{\omega^{(2)}_{i,j+1} \left( \frac{qdY^k_\lambda}{Y^{k+l}_{k+1}} \right) \cdot \omega^{(2)}_{i,j-1} \left( \frac{Y^k_\lambda}{qdY^{k+l}_{k+1}} \right)} \cdot \prod_{1 \leq i,j \leq k+l} (qd y_i - y_j)^2 \cdot C_{k+l} \right]$$

To simplify the notations, let $\omega^{(n)}_{i,j} \left( \frac{Y^k_\lambda}{qdY^{k+l}_{k+1}} \right) = \prod_{a \leq s \leq d} \omega^{(n)}_{i,s} \left( \frac{y_i}{y_j} \right)$, that is

$$\omega^{(2)}_{i,j} \left( \frac{qdY^k_\lambda}{qdY^{k+l}_{k+1}} \right) = \prod_{1 \leq i \leq k} (qd y_i - q^{-2} y_j) = \prod_{1 \leq i \leq k} \left( \frac{y_i - q^{-2} y_j}{y_i - y_j} \right)$$

$$\omega^{(2)}_{i,j} \left( \frac{Y^k_\lambda}{qdY^{k+l}_{k+1}} \right) = \prod_{1 \leq i \leq k} (y_i - q^2 y_j) = \prod_{1 \leq i \leq k} \left( \frac{y_i - q^2 y_j}{y_i - y_j} \right)$$

$$\omega^{(2)}_{i,j} \left( \frac{Y^k_\lambda}{Y^{k+l}_{k+1}} \right) = \prod_{1 \leq i \leq k} (y_i - q^2 y_j) = \prod_{1 \leq i \leq k} \left( \frac{y_i - q^2 y_j}{y_i - y_j} \right)$$

We can move the latter fraction of $\Upsilon_{2,1}(f \ast g)$ inside the $Sym$ bracket. After cancellation of common terms between the numerator and the denominator, we
get

\[ \Upsilon_{2,1}(f \ast g) = \text{Sym} \left[ \prod_{1 \leq i \leq k} (y_i - q^{2}y_j)(y_i - q^{2}d^2y_j)(y_i - d^{-2}y_j) \right] \frac{C_{k+l}}{C_k C_l} (qd)^{-2kl} \]

Picking \( c_k = (qd)^{k^2} \) or wherever, we get Formula for G follows:

\[ G = \prod_{1 \leq i < j \leq k} (qd^{1}y_i - y_j)^2 \frac{(qd)^{-k^2}}{\prod_{1 \leq i < j \leq k} (y_i - y_j)^2(y_i - q^2y_j)(y_i - q^{-2}y_j)} \]

with the parameters \( \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \) related to \( q, d \) via

\[ \tilde{q}_1 = d^2, \tilde{q}_2 = q^2, \tilde{q}_3 = q^{-2}d^{-2} \]

we construct an algebra homomorphism,

\[ \Upsilon_{2,1} : \bigoplus_{k} \text{Sym}_{(k,k)}(k,k)_{q_1,q_2,q_3} \to \text{Sym}_{(k_1)}(k)_{\tilde{q}_1 = q_1, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3} \]

\[ \Upsilon_{2,1} : f (X_k^k|Y_k^k) \to f (qdY_k^k|Y_k^k) \cdot \frac{(qd)^{-k^2} \prod_{1 \leq i < j \leq k} (qdy_i - y_j)^2}{\prod_{1 \leq i < j \leq k} (y_i - y_j)^2(y_i - q^2y_j)(y_i - q^{-2}y_j)} \]

**Remark 3.6** As we saw above, the verification of the fact that \( \Upsilon_{2,1} \) was an algebra homomorphism eventually boiled down to a solution for the system \( \frac{c_{k+l}}{c_k c_l} = \beta_{k,l} \) with the right-hand sides explicitly given. The existence of such \( c_k \) is guaranteed by the validity of the equalities \( \beta_{k+l,m} \beta_{k,l,m} = \beta_{k+l,m} \beta_{k,l,m} \) for any \( k, l, m \geq 1 \).

In the above example, we can easily see that \( c_k = (qd)^{-k^2} \) is a solution.

In the end, the existence of such constant is given by ??, but every constant is given as a fraction in the sentence below:

\[ \frac{c_{k+l+m}}{c_{k+l+m}} \cdot \frac{c_{k+l}}{c_k c_l} = \frac{c_{k+l+m}}{c_{k+l+m}} \cdot \frac{c_{l+m}}{c_l c_m} \]

(13)

### 3.6.1 Case 2: \( \lambda = qd^{-1} \)

We specialize \( X_k^k = qd^{-1}Y_k^k \).

Similar to the analysis of 3.5.1, we have

\[ G(Y_k^k) = \prod_{1 \leq i < j \leq k} \frac{(qd^{-1}y_i - y_j)^2}{(y_i - y_j)^2(y_i - q^2y_j)(y_i - q^{-2}y_j)} \cdot C_k \]

Again, we have to check whether the products are compatible and determine \( G(Y_k^k), \tilde{q}_1, \tilde{q}_2 \) and \( \tilde{q}_3 \).
with the parameters

$$\lambda = (q d)^{-1}$$

we construct an algebra homomorphism,

$$\Upsilon_{2,1}(f \cdot g) = \text{Sym} \left[ f(q d^{-2} Y^k_1 | Y^k_1) \cdot g(q d^{-2} Y^k_{1+k+1} | Y^{k+1}_{1+k+1}) \cdot \omega_{i,1}^{(2)} \left( \frac{Y^k_{1+k}}{q d^{-1} Y^k_{1+k+1}} \right) \right].$$

This time, we take $C_k = (q d^{-1})^{-k^2}$ then $\frac{C_k + 1}{C_k} (q d^{-2})^{|k|} = 1$. By picking

$$G(Y^k_1) = (q d^{-2})^{-k^2}. \prod_{1 \leq i,j \leq k} \frac{(q d^{-1} y_i - y_j)^2}{(y_i - y_j)^2} \prod_{1 \leq i < j \leq k} (y_i - q^2 d^{-1} y_j)(y_i - q^{-2} y_j)$$

with the parameters

$$\tilde{q}_1 = d^2 q^{-2}, \tilde{q}_2 = q^2, \tilde{q}_3 = d^{-2}$$

As explained in the beginning, there are only 4 possible choices for $\lambda$. The above are the cases that with $\lambda = q d$ in case 1 and $\lambda = q d^{-1}$ in case 2. To get to the remaining two cases, we just need swap the roles of $X^k_1$ and $Y^k_1$. We leave out the details and simply state the results:

### 3.6.2 Case 3 and Case 4

For $\lambda = q^{-1} d$:

$$\Upsilon_{2,1} : \bigoplus_k \text{Sh}_{q^{-2}} (k, k)|_{q_1, q_2, q_3} \rightarrow \text{Sh}_{q^{-2}} (k)|_{\tilde{q}_1 = q^{-1}, \tilde{q}_2 = q^{-2}, \tilde{q}_3 = q^{-1}}$$

given by

$$\Upsilon_{2,1} : f (X^k_1 | Y^k_1) \rightarrow f (q d^{-1} X^k_1 | Y^k_1) \cdot G(Y^k_1)$$

for $\lambda = q^{-1} d$. 

15
For $\lambda = q^{-1}d^{-1}$:

$$\Upsilon_{2,1} : \bigoplus_{k} S_{\text{gl}_2} \rightarrow S_{\text{gl}_1}$$

given by

$$\Upsilon_{2,1} : f (X^k_t | Y^k) \rightarrow f (X^k_t | qdX^k_t) \cdot G(X^k_t)$$

### 3.6.3 Summary of first example

Here, we shall point out that there are no other specializations because we have to map both $X^k_1$ and $Y^1_k$ to the same color, and we know that the quotient between their images must be our choices of $\lambda$, namely $a_{n,1}^{\pm 1}, a_{3}^{\pm 1}$ according to the wheel conditions. The last thing we need to check is whether $\Upsilon(f)$ satisfies the wheel conditions of $S_{\text{gl}_1}$. This is clear as we examine the variables in $Y^1_k$ which is showed in figure $5$. For example, as to case $1$, where $\lambda = qd$. If $S_{\text{gl}_1}$ wheel in color $(0, 0, 1)$, then there exist $x_a, y_b$ and $z_c$, such that $y_b = qd$, and $x_c = qd^{-1}y_b$. Since $x_i = qd$, let’s assume $y_a = t$, then $x_a = qd^2, y_b = q^2d^2, y_c = q^2t$, therefore, $y_b = qd^2y_a, y_c = q^2d^{-1}y_b$, which is exactly the wheel conditions for $S_{\text{gl}_1}$.

Here, we have established the simplest case of $n = 2$ of Theorem $3.2.

![Figure 5: Check of wheel conditions](image)

### 3.7 From $S_{\text{gl}_2}$ to $S_{\text{gl}_1}$

We utilize the same idea as explained in the section $3.4$ to construct the homomorphism $\Upsilon_{3,2} : S_{\text{gl}_2} \rightarrow S_{\text{gl}_1}$. In $S_{\text{gl}_2}$, we have variables of $X_{1,1}^{1, k}, X_{2,1}^{2, k}, X_{3,1}^{3, n}$ in three different colors. To simplify notations, we rename $X_{1,1}^{1, k}, X_{2,1}^{2, k}$ and $X_{3,1}^{3, n}$ by $X^k_1, Y^k_1$ and $Z^n_1$ respectively and set $X^k_1$ as color $0$, $Y^k_1$ as color $1$ and $Z^n_1$ as color $2$. Since $X^k_1$ and $Y^k_1$ have the same dimension, according to the analysis in section $3.4$ on constructing algebra homomorphism from $S_{\text{gl}_2}$ to $S_{\text{gl}_1}$, we specialize $Y^1_k$ to $\lambda X^k_1$ with $\lambda \in \{q_1^{-1}, q_3\}$ and $Z^n_1$ to $\mu Z^n_1$. Therefore, the map $\Upsilon_{3,2}$ is of the following form:

$$\Upsilon_{3,2} : f (X^k_1 | Y_1^k | Z^n_1) \rightarrow f (X^k_1 | \lambda X^k_1 | \mu Z^n_1) \cdot G(X^k_1 | Z^n_1)$$

Our goal is to determine the rational factor $G(X^k_1 | Z^n_1)$, as well as $\lambda$ and $\mu$ in terms of $q, d$, and get the explicit relation between $q_1, q_2, q_3$ and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ so
that the map of $\Upsilon_{3,2}$ is compatible with the shuffle product. By utilizing the wheel conditions, we find that $\lambda \in \{q^{-1}d^{-1}, qd^{-1}\}$.

### 3.7.1 case 1: $\lambda = q_3 = q^{-1}d^{-1}$

We are looking for a map of the following form:

$$
\Upsilon_{3,2} : f \left( X^i_1 | Y^k_1 | Z^n_1 \right) \to f \left( X^i_1 | q^{-1}d^{-1}X^k_1 | \mu Z^\mu_1 \right) \cdot G \left( X^i_1 | Z^n_1 \right)
$$

In order to find the rational factor of $G \left( X^i_1 | Z^n_1 \right)$ that ensure $\Upsilon_{3,2}$ is an algebra homomorphism, we need to analyze the conditions that $G$ needs to meet. The numerator of $G$ should be able to clear off the denominator of $f \left( X^i_1 | q^{-1}d^{-1}X^k_1 | \mu Z^\mu_1 \right)$. In addition, the denominator of $G$ should create the desired pole involving $\prod_{1 \leq j \leq k} (x_i - z_j)$ in $Sh = (k, n)$. Finally, it should clear off zeroes due to the extra wheel conditions of $Sh = (k, n)$. Based on the above analysis, we have:

$$
G \left( X^i_1 | Z^n_1 \right) = \frac{\prod_{1 \leq j \leq k} (x_i - q^{-1}d^{-1}x_j) \prod_{1 \leq i \leq k} (q^{-1}d^{-1}x_i - \mu z_j)(\mu z_j - x_i)}{\prod_{1 \leq j \leq k} (x_i - z_j)^2 \prod_{1 \leq c < j \leq k} (x_i - q^2x_j)(x_i - q^{-2}x_j)} \cdot C_{k,n}
$$

Let $f \left( X^i_1 | Y^k_1 | Z^n_1 \right) \in Sh = (k, n)$ and $g \left( X^{k+l}_{k+1} | Y^{k+l}_{k+1} | Z^{n+m}_{n+1} \right) \in Sh = (l, l, m)$. The choice of $\lambda, \mu$ and $G$ should ensure that the equation (14) holds.

$$
\omega^{(3)} \left( \frac{X^i_1 | \lambda X^k_1 | \mu Z^\mu_1}{X^i_1 \lambda X^k_1 \mu Z^\mu_1} \right) = \frac{G \left( X^{k+l}_{k+1} | Z^{n+m}_{n+1} \right)}{G \left( X^i_1 | Z^n_1 \right) \cdot G \left( X^{k+l}_{k+1} | Z^{n+m}_{n+1} \right)} \cdot \omega^{(2)} \left( \frac{X^i_1 | \lambda X^k_1 | \mu Z^\mu_1}{X^i_1 \lambda X^k_1 \mu Z^\mu_1} \right) \quad (14)
$$

Let’s compute each of the factors in the equation (14):

$$
\omega^{(3)} \left( \frac{X^i_1 | \lambda X^k_1 | \mu Z^\mu_1}{X^{k+l}_{k+1} | \lambda X^{k+l}_{k+1} | \mu Z^{n+m}_{n+1}} \right) = \omega^{(3)} \left( \frac{X^i_1}{X^{k+l}_{k+1}} \right) \cdot \omega^{(3)} \left( \frac{\lambda X^k_1}{\lambda X^{k+l}_{k+1}} \right) \cdot \omega^{(3)} \left( \frac{\mu Z^\mu_1}{\mu Z^{n+m}_{n+1}} \right)
$$

$$
\omega^{(3)} \left( \frac{X^i_1}{X^{k+l}_{k+1}} \right) \cdot \omega^{(3)} \left( \frac{\lambda X^k_1}{\mu Z^{n+m}_{n+1}} \right) \cdot \omega^{(3)} \left( \frac{\mu Z^\mu_1}{X^{k+l}_{k+1}} \right)
$$

$$
\omega^{(3)} \left( \frac{\lambda X^k_1}{X^{k+l}_{k+1}} \right) \cdot \omega^{(3)} \left( \frac{\mu Z^\mu_1}{X^{k+l}_{k+1}} \right)
$$

$$
\omega^{(3)} \left( \frac{\mu Z^\mu_1}{X^{k+l}_{k+1}} \right)
$$

17
That is,

\[ \omega^{(3)} \left( \frac{X_k^{i+1} | \lambda X_k^i | \mu Z_k^m}{X_k^{i+1} | \lambda X_k^i | \mu Z_{n+1}^m} \right) = \prod_{1 \leq i \leq k} \left( \frac{x_i - q d^{-1} \mu z_r - d^{-2} q^{-1} x_i - q \mu z_r}{x_i - \mu z_r - d^{-1} q^{-1} x_i - \mu z_r} \right) \]

\[ k < j \leq k+l \]

\[ \prod_{1 \leq i \leq k} \left( \frac{x_i - q^{-2} x_j}{x_i - x_j} \cdot \frac{d^{-1} (x_i - x_j)}{x_i - x_j} \right) \cdot \prod_{1 \leq r \leq n} \left( \frac{z_r - q^{-2} z_s}{z_r - z_s} \right) \]

\[ n < r \leq n+m \quad (15) \]

\[ \{ \tilde{q} d^{-1}, \tilde{q} d \} = \{ q d^{-1} \mu, q^2 d^2 \mu \} \]

Comparing terms of \((z_r, z_s)\) in the equation of \((15),(16)\) and \((17)\) we get \(\tilde{q} = q\).

Comparing terms of \((x_i, z_r)\) with \(1 \leq i \leq k, n < r \leq n + m\), in the equation of \((15),(16)\) and \((17)\), we get condition:

\[ \{ \tilde{q} d^{-1}, \tilde{q} d \} = \{ q d^{-1} \mu, d^{-2} \mu^{-1} \} \]

As far as condition \((18)\) is concerned,

\[ \tilde{q} d^{-1} \cdot \tilde{q} d = q d^{-1} \mu \cdot q^2 d^2 \mu \Rightarrow \tilde{q}^2 = q^3 d \mu^2 \Rightarrow \mu = (q d)^{-1/2} \]

When \(\mu = (q d)^{-1/2}\), we have \(q d \mu^{-1} = (q d)^{3/2} = q^2 d^2 \mu\) and \(d^{-2} \mu^{-1} = q^{1/2} d^{-3/2} = q d^{-1} \mu\). Therefore, \((18)\) and \((19)\) are compatible, and we also have \(\{ d \pm 1 \} \)

The remaining thing to be verified is that the image satisfies wheel condition of \(Sh_{\tilde{g}_2^\pm} (k, n)\).
1. The $\text{Sh}_{\varnothing^k_1}$ wheel in color $(0,0,2)$ implies vanishing as $\frac{\mu}{\mu^{\mu}} = \frac{q}{d}$, $\frac{\mu^2}{\mu^{\mu}} = qd$ for some $1 \leq a, b \leq k, 1 \leq c \leq n$. Now $\mu = (qd)^{-1/2}$, we have $\frac{\mu}{\mu^{\mu}} = qd^{-\frac{1}{2}} = \tilde{q}d^{-1}$ and $\frac{\mu^2}{\mu^{\mu}} = q^2d^{-\frac{3}{2}} = \tilde{q}d^{-1}$ which is exactly one of the wheel conditions for $\text{Sh}_{\varnothing^k_1}$.

2. The $\text{Sh}_{\varnothing^k_2}$ wheel in color $(1,1,2)$ implies vanishing as $\frac{\mu^{-1}d^{-1}x}{\mu^{\mu}} = qd$, and $\frac{\mu^{-1}d^{-1}x}{q^{-d}x} = \frac{q}{d}$. We have $\frac{\mu}{\mu^{\mu}} = q^{-\frac{1}{2}}d^{-\frac{1}{2}} = \tilde{q}d^{-1}$ which is exactly one of the wheel conditions for $\text{Sh}_{\varnothing^k_2}$.

3. The $\text{Sh}_{\varnothing^k_3}$ wheel in $(2,2,0)$ and $(2,2,1)$ are checked in the same way.

A direct computation shows that the constant $c_k, n$ can be nicely incorporated into $G$, by replacing $G$ with $\tilde{G}$:

$$
\tilde{G}(X^k_i|Z^n_i) = \frac{\prod_{1 \leq i,j \leq k} (dx_i - d^2q x_j)}{\prod_{1 \leq i<j \leq k} (x_i - q^2 x_j)(x_i - q^{-2} x_j)} 
\prod_{1 \leq i \leq n} (x_i - q^{-\frac{1}{2}}d z_i) \prod_{1 \leq j \leq n} (x_j - q^{-\frac{1}{2}}d z_j) .
$$

with parameters $\tilde{q}_1 = q^{-\frac{1}{2}}d^{\frac{1}{2}}, \tilde{q}_2 = q^2, \tilde{q}_3 = q^{-\frac{1}{2}}d^{\frac{3}{2}}$

we have constructed an algebra homomorphism:

$$
\Upsilon_{3.2} : \bigoplus_{k,n} \text{Sh}_{\varnothing^k_1}(k,k,n)|_{q_1,q_2,q_3} \rightarrow \text{Sh}_{\varnothing^k_2}(k,n)|_{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3} \quad \text{given by}
$$

$$
\Upsilon_{3.2} : f(X^k_i|Y^k_i|Z^n_i) \rightarrow f(X^k_i|q^{-1}d^{-1}X^k_i|q^{-\frac{1}{2}}d^{-\frac{3}{2}} Z^n_i) \cdot \tilde{G}(X^k_i|Z^n_i) .
$$

In addition to $Y^k_i = q^{-1}d^{-1}X^k_i$, we have three other specializations to consider which are $Y^k_i = q^{-1}dX^k_i$, $Y^k_i = qdX^k_i$ and $Y^k_i = qd^{-1}X^k_i$.

**3.7.2 case 2:** $\lambda = q^{-1} = qd^{-1}$

When $Y^k_i = qd^{-1}X^k_i$, we are looking for a map of the following form:

$$
\Upsilon_{3.2} : f(X^k_i|Y^k_i|Z^n_i) \rightarrow f(X^k_i|qd^{-1}X^k_i|\mu Z^n_i) \cdot G(X^k_i|Z^n_i)
$$

The rational factor of $G$ is:

$$
G(X^k_i|Z^n_i) = \frac{\prod_{1 \leq i,j \leq n} (x_i - qd^{-1} x_j) \prod_{1 \leq j \leq n} (qd^{-1} x_i - \mu x_j)(\mu x_j - x_i) \prod_{1 \leq i \leq n} (x_i - z_i)^2 \prod_{1 \leq i,j \leq k} (x_i - q^2 x_j)(x_i - q^{-2} x_j)}{\prod_{1 \leq i \leq n} (x_i - z_i)^2 \prod_{1 \leq i,j \leq k} (x_i - q^2 x_j)(x_i - q^{-2} x_j)} \cdot C_{k,n}
$$
For $f \left( X^k | Y^k | Z^n \right) \in Sh_{\mathbb{P}^3} (k, k, n)$ and $g \left( X^{k+1} | Y^{k+1} | Z^{n+m} \right) \in Sh_{\mathbb{P}^3} (l, l, m)$

$$
\omega^{(3)} \left( X^k | \lambda X^k | [\mu Z]^n \right) = \prod_{1 \leq i \leq k} \left( x_i - q^{-2} x_j \right) \cdot \frac{d^{-1} x_i - q^{d-1} x_j}{x_i - x_j} \cdot \frac{x_i - q^{-2} x_j}{x_i - x_j} \cdot \frac{q^{d-1} x_i - q^{d-1} x_j}{q^{d-1} x_i - q^{d-1} x_j},
$$

$$
\prod_{1 \leq i \leq k} \frac{x_i - q^{d-1} x_j}{x_i - x_j} \cdot \frac{q^{d-1} x_i - q^{d-1} x_j}{x_i - x_j}.
$$

The remaining thing to be verified is that the image satisfies wheel condition of $G$:

$$
G \left( X^k | Y^k | Z^n \right) G \left( X^{k+1} | Y^{k+1} | Z^{n+m} \right) = \frac{C_{k+l+n+m}}{C_{k,m} C_{l,m}},
$$

$$
\prod_{1 \leq i \leq k} \frac{x_i - q^{d-1} x_j}{x_i - x_j} \cdot \frac{q^{d-1} x_i - q^{d-1} x_j}{x_i - x_j}.
$$

By taking the product of above two factors and comparing the terms of $(z_r, z_s)$, $(x_i, x_j)$, $(x_i, z_r)$, and $(x_r, z_s)$, with the corresponding item in

$$
\omega^{(2)} \left( X^k | \lambda X^k | [\mu Z]^n \right),
$$

we have $\mu = q_1 - 1/2$, $\tilde{q}_1 = q_1^{1/2}$, $\tilde{q}_2 = q_2$, $\tilde{q}_3 = \mu q_1 - 1/2$.

The remaining thing to be verified is that the image satisfies wheel condition of $Sh_{\mathbb{P}^3} (k, n)$.

1. $Sh_{\mathbb{P}^3}$ wheel in color $(0, 0, 2)$ implies vanishing when $\frac{x_a}{\mu z_c} = \frac{q^{2} d^{-2}}{2}$, $\frac{z_a}{x_b} = \frac{q^{2} d^2}{2}$ which is also equivalent to $\frac{x_a}{x_b} = \tilde{q}_d^{-1}$, $\frac{z_a}{z_b} = \tilde{q}_d^{-1}$.

2. $Sh_{\mathbb{P}^3}$ wheel in color $(1, 1, 2)$ implies vanishing when $\frac{q^{d-1} x_a}{\mu z_c} = q d$, $\frac{z_a}{x_b} = \frac{q^{2} d^2}{2}$ which is also equivalent to $\frac{x_a}{x_b} = \tilde{q}_d^{2}$, $\frac{z_a}{z_b} = \tilde{q}_d^{2}$.

3. $Sh_{\mathbb{P}^3}$ wheel in color $(2, 2, 0)$ implies vanishing when $\frac{\mu z_a}{x_c} = q d$, $\frac{x_a}{\mu z_b} = \frac{q}{2}$ which is also equivalent to $\frac{z_a}{x_c} = q^{2} d^{2}$, $\frac{x_a}{z_b} = q^{2} d^{-2}$.

4. $Sh_{\mathbb{P}^3}$ wheel in color $(2, 2, 1)$ implies vanishing when $\frac{\mu z_a}{q d^{-1} x_c} = q d^{-1}$, $\frac{q^{d-1} x_a}{\mu z_b} = q d$ which is also equivalent to $\frac{x_a}{x_c} = q^{2} d^{-2}$, $\frac{z_a}{z_b} = q^{2} d^{2}$.
By straightforward computation, we can incorporate the constant $c_{k,n}$ in $G$ by replacing $G$ with $\tilde{G}$, and we will get algebra homomorphism:

$$\Upsilon_{3,2} : \bigoplus_{k,n} Sh_{\mathfrak{g} \mathfrak{t}_1} (k, n)_{q_1, q_2, q_3} \rightarrow Sh_{\mathfrak{g} \mathfrak{l}_2} (k, n)_{\frac{q_1}{q_1}, \frac{q_2}{q_2}, \frac{q_3}{q_3}, \frac{q_4}{q_4}}$$

given by

$$\Upsilon_{3,2} : f \left( X^n_Y | Y^n_X | Z^n_Y \right) \rightarrow f \left( X^n_{Y|Y^n} | q^n d^{-1} X^n_Y | \frac{\tilde{G}}{\tilde{G}} \cdot \tilde{G} \left( X^n_{Y|Z^n_Y} \right) \right)$$

with

$$\tilde{G} \left( X^n_{Y|Z^n_Y} \right) = \prod_{1 \leq i,j \leq k} (x_i - q^{-1} x_j) \prod_{1 \leq i \leq k} (x_i - q^2 x_j) \prod_{1 \leq i \leq k} (x_i - q^{-2} x_j) \prod_{1 \leq r \leq n} \left( d^{-1} q^n x_i - z_r \right) \left( d^{-1} q^{-n} x_i - z_r \right) \left( x_i - z_r \right)^2$$

In summary, we have two homomorphism of $\Upsilon_{3,2}$, which are:

$$\Upsilon_{3,2} : f \left( X^n_Y | Y^n_X | Z^n_Y \right) \rightarrow f \left( X^n_{Y|Y^n} | q^n d^{-1} X^n_Y | \frac{\tilde{G}}{\tilde{G}} \cdot \tilde{G} \left( X^n_{Y|Z^n_Y} \right) \right)$$

$$\Upsilon_{3,2} : f \left( X^n_Y | Y^n_X | Z^n_Y \right) \rightarrow f \left( X^n_{Y|Y^n} | q^n d^{-1} X^n_Y | \frac{\tilde{G}}{\tilde{G}} \cdot \tilde{G} \left( X^n_{Y|Z^n_Y} \right) \right)$$

### 3.8 From $Sh_{\mathfrak{g} \mathfrak{l}_3}$ to $Sh_{\mathfrak{g} \mathfrak{l}_1}$

We can construct the homomorphism $\Upsilon_{3,1}$ from $Sh_{\mathfrak{g} \mathfrak{l}_3}$ to $Sh_{\mathfrak{g} \mathfrak{l}_1}$ in two ways. One is a two-step method. That is, we firstly construct the homomorphisms $\Upsilon_{3,2}$ from $Sh_{\mathfrak{g} \mathfrak{l}_3}$ to $Sh_{\mathfrak{g} \mathfrak{l}_2}$, then construct the homomorphisms $\Upsilon_{2,1}$ from $Sh_{\mathfrak{g} \mathfrak{l}_2}$ to $Sh_{\mathfrak{g} \mathfrak{l}_1}$, and finally compose two homomorphisms $\Upsilon_{3,2}$ and $\Upsilon_{2,1}$ together. The other way is a one-step method, which is similar to what we have done in constructing the homomorphism from $Sh_{\mathfrak{g} \mathfrak{l}_3}$ to $Sh_{\mathfrak{g} \mathfrak{l}_1}$.

In this section, we chose the latter. We specify $X^n_Y$ color 0, $Y^n_Y$ color 1 and $Z^n_Y$ color 2. We propose the following:

**Proposition 3.9** There are four ways to map $Sh_{\mathfrak{g} \mathfrak{l}_3}$ to $Sh_{\mathfrak{g} \mathfrak{l}_1}$ via a component-wise in the general form: $\Upsilon_{3,1} : f \left( X^n_Y | Y^n_X | Z^n_Y \right) \rightarrow f \left( X^n_{Y|X^n} | \lambda X^n_Y | \lambda \mu X^n_Y \right) \cdot G \left( X^n \right)$ for $\lambda, \mu \in \{ q_3, q_4^{-1} \}$.

We will provide the more specific constructions of the four mappings in the case studies bellow:

#### 3.9.1 Case 1:

$\lambda = \mu = q_3 = q^{-1}d^{-1}$

In this case, we specialize $Y^n_Y = q^{-1}d^{-1}X^n_Y$, and $Z^n_Y = q^{-2}d^{-2}X^n_Y$.

The factor of $G \left( X^n_Y \right)$ should meet following conditions:

1. The numerator of $G \left( X^n_Y \right)$ should be able to clear off the denominator of $f \left( X^n_{Y|X^n} | \lambda X^n_Y | \lambda \mu X^n_Y \right)$.
2. The denominator of $G(X^k_1)$ should create the desired pole in $Sh_{q^i}$, and clear off zeroes due to wheel conditions. Based on the above analysis, we have:

$$G(X^k_1) = \prod_{1 \leq i \leq k} \left( x_i - q^{-1}d^{-1}x_j \right) (q^{-1}d^{-1}x_i - q^{-2}d^{-2}x_j) (q^{-2}d^{-2}x_i - x_j)$$

In order to determine the explicit relation between $q_1, q_2, q_3$ and $q_1, q_2$, let $f(X^k_1|Y^k_i|Z^k_i) \in Sh_{q^i}$, and $g(X^{k+1}_1|Y^{k+1}_1|Z^{k+1}_1) \in Sh_{q^i}$.

We write down the shuffle factors:

$$\omega^{(3)} \left( \frac{X^k_i|\lambda^k_i|\lambda^k_i X^k_i}{X^{k+1}_i|\lambda^{k+1}_i|\lambda^{k+1}_i X^{k+1}_i} \right) = \prod_{1 \leq i \leq k} \left( x_i - q^{-2}x_j \right) \left( x_i - q^{-2}x_j \right) \left( x_i - q^{-2}x_j \right) \left( x_i - x_j \right),$$

$$\prod_{1 \leq i \leq k} \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - x_j \right),$$

$$\prod_{1 \leq i \leq k} \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - x_j \right).$$

The ratio of $\frac{G(X^{k+1}_1)}{G(X^k_1)}$, is equal to:

$$\prod_{1 \leq i \leq k} \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - x_j \right),$$

The product of the above two factors is:

$$\omega^{(3)} \left( \frac{X^k_i|\lambda^k_i|\lambda^k_i X^k_i}{X^{k+1}_i|\lambda^{k+1}_i|\lambda^{k+1}_i X^{k+1}_i} \right) = \frac{G(X^{k+1}_1)}{G(X^k_1)} \cdot \prod_{1 \leq i \leq k} \left( x_i - q^{-2}x_j \right) \left( x_i - q^{-1}d^{-1}x_j \right) \left( x_i - x_j \right)^3.$$
Thus, we have constructed a homomorphism
\[ G \]
In this case, we specialize
\[ \lambda = q^{-1} \] 3.9.2 Case 2: \[ G \]
We write down the shuffle factors:
\[ \omega(3) \] 3. Other cases can be verified similarly.

Thus, we have constructed a homomorphism \( \Upsilon_{3,1} \)
\[ \Upsilon_{3,1} : \bigoplus_k Sh_{q_{13}} (k, k, k) \to Sh_{q_{13}} (k) \] given by
\[ \Upsilon_{3,1} : f (X_1^k \vert Y_1^k \vert Z_1^k) \to f (X_1^k \vert q^{-1} d^{-1} \vert Y_1^k \vert q^{-2} d^{-2} X_1^k) \cdot G (X_1^k) \] with
\[ G (X_1^k) = \prod_{1 \leq i,j \leq k} (x_i - q^{-1} d^{-1} x_j) (q^{-1} d^{-1} x_i - q^{-2} d^{-2} x_j) \]
\[ \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 (x_i - q^2 x_j)^2 \]
\[ \prod_{1 \leq i,j \leq k} (q^{-2} d^{-2} x_i - x_j) \]
\[ \prod_{1 \leq i < j \leq k} (x_i - q^{-2} x_j)^2 \cdot (-q^3 d^3)^k \]

The existence of such constant \( c_k \) is due to the criteria of Remark 3.6.

3.9.2 Case 2: \( \lambda = \mu = q_{13}^{-1} = q^{-1} \)
In this case, we specialize \( Y_1^k = q^{-1} d^{-1} X_1^k \) and \( Z_1^k = q^{-2} d^{-2} X_1^k \)
The \( G (X_1^k) \) should be of the form:
\[ G (X_1^k) = \prod_{1 \leq i,j \leq k} (x_i - q^{-1} d^{-1} x_j) (q^{-1} d^{-1} x_i - q^{-2} d^{-2} x_j) (q^{-2} d^{-2} x_i - x_j) \]
\[ \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 (x_i - q^2 x_j)^2 (x_i - q^{-2} x_j)^2 \cdot C_k \]

We write down the shuffle factors:
\[ \omega^{(3)} \left( \frac{X_1^k \vert X_1^k \vert \mu X_1^k}{X_{k+1}^k \vert X_{k+1}^k \vert \mu X_{k+1}^k} \right) = \prod_{1 \leq i \leq k} \left( \frac{x_i - q^{-2} x_j}{x_i - x_j} \right)^3 \]
\[ \prod_{1 \leq i,j \leq k} \left( \frac{q^{-2} d^{-2} x_i - q^{-3} d^{-2} x_j}{q^{-1} d^{-1} x_i - q^{-2} d^{-2} x_j} \right) \cdot \left( \frac{q^{-1} d^{-1} x_i - q^{-4} d^{-2} x_j}{q^{-2} d^{-2} x_i - x_j} \right) \cdot \left( \frac{x_i - q^{-3} d^{-3} x_j}{x_i - q^{-2} d^{-2} x_j} \right) \]
\[ \prod_{1 \leq i,j \leq k} \left( \frac{q^{-1} d^{-1} x_i - q^{-2} d^{-1} x_j}{q^{-1} d^{-1} x_i - x_j} \right) \cdot \left( \frac{q^{-2} d^{-2} x_i - q^{-4} d^{-2} x_j}{q^{-2} d^{-2} x_i - x_j} \right) \cdot \left( \frac{q^{-2} d^{-2} x_i - q^{-2} d^{-1} x_j}{x_i - q^{-2} d^{-1} x_j} \right) \]
The ratio of \( \frac{G(X^{k+1})}{G(X^k)G(X^l)} \) is equal to:

\[
\prod_{1 \leq i \leq k}^{k < j \leq k+l} \left( \frac{(x_i - qd^{-1}x_j)(qd^{-1}x_i - q^2d^{-2}x_j)(q^2d^{-2}x_i - x_j)}{(x_i - x_j)^3} \right).
\]

The product of the above two factors is:

\[
\omega^{(3)} \left( \frac{X^k_1|X^k_1|\lambda \mu X^k_1}{X^k_{k+1}|X^k_{k+1}|\lambda \mu X^k_{k+1}} \right) \cdot \frac{G(X^{k+1})}{G(X^k)G(X^l)} = \prod_{1 \leq i \leq k}^{k < j \leq k+l} \left( \frac{(x_i - q^{-2}x_j)(x_i - q^{-1}d^3x_j)(x_i - q^3d^{-3}x_j)}{(x_i - x_j)^3} \right) \cdot \frac{G_{k+l}}{G_kG_l}.
\]

The last thing to be verified is that the image satisfies wheel condition of \( Sh_{\overline{\theta}_i} \).

1. The \( Sh_{\overline{\theta}_i} \) wheel in color \((0,0,2)\) implies vanishing as \(\frac{x_i}{x_k} = \frac{\partial qd}{\partial x_k} = qd\) for some \(1 \leq a, b, c \leq k\), which means \(\frac{x_i}{x_k} = \partial_3 = \partial_3\), \(\frac{x_i}{x_k} = q^{-1}d^3 = \partial_3^{-1}\).

2. The \( Sh_{\overline{\theta}_i} \) wheel in color \((2,2,0)\) implies vanishing as \(\frac{q^2d^{-2}x_k}{x_k} = qd\), \(\frac{x_i}{x_k} = \frac{d_3}{q} = \partial_3^{-1} \), \(\frac{x_i}{x_k} = q^3d^{-3} = \partial_3^{-1}\).

Thus, we have constructed an algebra homomorphism \( \mathcal{Y}_{3.1} : \bigoplus_k Sh_{\overline{\theta}_i} (k,k,k) \rightarrow Sh_{\overline{\theta}_i} (k)|_{\partial_3 = \partial_3^2 = \partial_3^3 = \partial_3^{-2}} \) given by

\[
\mathcal{Y}_{3.1} : f(X^k_1|X^k_1|Z^k_1) \rightarrow f(X^k_1|qd^{-1}X^k_1|q^{-2}d^{-2}X^k_1) \cdot G(X^k_1)
\]

with

\[
G(X^k_1) = \prod_{1 \leq i \leq j \leq k} \left( x_i - qd^{-1}x_j \right)^2 \left( x_i - q^2d^{-2}x_j \right) \left( x_i - q^{-1}d^3x_j \right) \left( x_i - q^3d^{-3}x_j \right)^2
\]

**3.9.3 Case 3:** \( \lambda = q_3 = q^{-1}d^{-1}, \mu = q_1^{-1} = qd^{-1} \)

In this case, we specialize \( Y^k_1 = q^{-1}d^{-1}X^k_1 \) and \( Z^k_1 = d^{-2}X^k_1 \).

The \( G(X^k_1) \) should be of the form:

\[
G(X^k_1) = \prod_{1 \leq i \leq j \leq k} \left( x_i - q^{-1}d^{-1}x_j \right)^2 \left( x_i - d^{-2}x_j \right) \left( x_i - d^{-2}x_j \right) \left( x_i - q^2d^{-2}x_j \right) \left( x_i - q^3d^{-3}x_j \right)^2 \cdot C_k
\]

24
We write down the specializations of shuffle factors:

\[ \omega^{(3)} \left( \frac{\lambda^X | \lambda X_k | \mu X_{k+1}}{X_{k+1} | \lambda^X_{k+1} | \mu X_{k+1+1}} \right) = \prod_{1 \leq i \leq k} \left( \frac{x_i - q^{-2}x_j}{x_i - x_j} \right)^3, \]

\[ \prod_{1 \leq i \leq k} \left( \frac{q^{-1}d^2x_i - qdx_j}{q^{-1}d^2x_i - dx_j} \cdot \frac{d^2x_i - q^2x_j}{d^2x_i - dx_j} \right). \]

The ratio of \( \frac{G(X_{k+1})}{G(X_k)} \) is equal to:

\[ \prod_{1 \leq i \leq k} \left( \frac{x_i - q^{-1}d^2x_i}{x_i - x_j} \right)^2. \]

The product of the above two factors is:

\[ \omega^{(3)} \left( \frac{\lambda^X | \lambda X_k | \mu X_{k+1}}{X_{k+1} | \lambda^X_{k+1} | \mu X_{k+1+1}} \right) \cdot \frac{G(X_{k+1})}{G(X_k)G(X_1)} = \prod_{1 \leq i \leq k} \left( \frac{x_i - q^{-2}x_j}{x_i - x_j} \right)^3. \]

The last thing to be verified is that the image satisfies wheel condition of \( Sh_{\tilde{g}_1} \).

1. The \( Sh_{\tilde{g}_1} \) wheel in color \((0, 0, 2)\) implies vanishing as \( \frac{x_a}{q} = qd^{-2} = \frac{x_b}{x_c} = qd \) for some \( 1 \leq a, b, c \leq k \), which means \( \frac{x_a}{x_b} = qd^{-3} = qd^{-1} \).

2. The \( Sh_{\tilde{g}_2} \) wheel in color \((2, 2, 0)\) implies vanishing as \( \frac{d^2x_a}{x_b} = qd, \frac{x_a}{x_b} = qd^{-1} \), which means \( \frac{x_a}{x_b} = qd^3 = \tilde{q}_1 \).

Thus, we have constructed a possible algebra homomorphism

\[ \Upsilon_{3.1} : \bigoplus_{\tilde{g}_1} Sh_{\tilde{g}_1} (k, k, k) | q_{1, q_{2, q_3}} \rightarrow Sh_{\tilde{g}_1} (k) | q_{1, q_{2, q_3}} = q_1^q q_2^{-1} q_2 = q_3^{-1} q_3^q \] given by

\[ \Upsilon_{3.1} : f (X_{k+1} | Y_1^2 | Z_k) \rightarrow f (X_{k+1} | q^{-1}d^2X_i | d^2X_k) \cdot G (X_k) \] with

\[ G (X_k) = \prod_{1 \leq i < j \leq k} \left( \frac{x_i - x_j}{x_i - q^2x_j} \right)^2 \left( x_i - q^{-2}x_j \right)^2. \]
3.9.4 Case 4: \( \lambda = q_1^{-1} = qd^{-1}, \mu = q_3 = q^{-1}d^{-1} \)

In this case, we specialize \( Y_1^k = qd^{-1}X_1^k \) and \( Z_1^k = d^{-2}X_1^k \)

The \( G(X_1^k) \) should be of the form:

\[
G(X_1^k) = \frac{\prod_{1 \leq i < j \leq k} (x_i - qd^{-1}x_j)(qd^{-1}x_i - d^{-2}x_j)(d^{-2}x_i - x_j)}{\prod_{1 \leq i < j \leq k} (x_i - x_j)^2(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2} \cdot C_k
\]

We write down the specializations of shuffle factors:

\[
\omega^{(3)} \left( \frac{X_1^k | \lambda X_1^k | \lambda \mu X_1^k}{X_1^{k+1} | \lambda X_1^{k+1} | \lambda \mu X_1^{k+1}} \right) = \prod_{1 \leq i < j \leq k} \left( \frac{x_i - qd^{-2}x_j}{x_i - x_j} \right)^3.
\]

The ratio \( \frac{G(X_1^{k+1})}{G(X_1^k)G_i} \) is equal to:

\[
\prod_{1 \leq i < j \leq k} \left( x_i - qd^{-1}x_j \right) (qd^{-1}x_i - d^{-2}x_j) (d^{-2}x_i - x_j)
\]

The product of the above two factors is:

\[
\omega^{(3)} \left( \frac{X_1^k | \lambda X_1^k | \lambda \mu X_1^k}{X_1^{k+1} | \lambda X_1^{k+1} | \lambda \mu X_1^{k+1}} \right) \cdot G(X_1^{k+1}) = \prod_{1 \leq i < j \leq k} \left( \frac{x_i - q^{-2}x_j}{x_i - x_j} \right)^3 \frac{(x_i - qd^3x_j)(x_i - qd^3x_j)(x_i - q^{-2}x_j)}{(x_i - x_j)^3} .
\]

The last thing to be verified is that the image satisfies wheel condition of \( Sh_{q_1} \).

Since \( \tilde{q}_1, \tilde{q}_2 \) and \( \tilde{q}_3 \) is equal to those in case 3, they also satisfy the wheel conditions. Thus, we have constructed a possible algebra homomorphism:

\[
\mathcal{H}_{3.1} : \bigoplus_k Sh_{q_1} (k, k, k) \rightarrow Sh_{q_1} (k) |_{\tilde{q}_1 = q_1^{-1}q_1^{-1}, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3}
\]
In summary, we have constructed 4 homomorphisms of the following form:

\[ \Upsilon_{\alpha,\beta} : \bigoplus_k \text{Sh}_{\mathfrak{gl}_3} (k, k, |q_1, q_2, q_3) \to \text{Sh}_{\mathfrak{gl}_1} (k) \bigg| \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \]

\[ \Upsilon_{\alpha,\beta} : f (X^k | Y^i | Z^k) \to f (X^k | \lambda X^k | \mu X^k) \cdot G (X^k) \]

with the following values of \( \lambda, \mu \) and \( \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \):

1. \( \lambda = q^{-1}d^{-1}, \mu = q^{-1}d^{-1} \)
   \( \tilde{q}_1 = q_1q_3^{-2}, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3^3 \)
   \( \tilde{q} = q, \tilde{d} = q^2d^3 \)

2. \( \lambda = qd^{-1}, \mu = qd^{-1} \)
   \( \tilde{q}_1 = q^3q_3^{-2}, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3^2q_3^{-1} \)
   \( \tilde{q} = q, \tilde{d} = q^{-2}d^3 \)

3. \( \lambda = q^{-1}d^{-1}, \mu = qd^{-1} \)
   \( \tilde{q}_1 = q_1^2q_3^{-2}, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3^2q_3^{-1} \)
   \( \tilde{q} = q, \tilde{d} = d^3 \)

4. \( \lambda = qd^{-1}, \mu = q^{-1}d^{-1} \)
   \( \tilde{q}_1 = q_1q_3^3, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3^3q_3^{-1} \)
   \( \tilde{q} = q, \tilde{d} = d^3 \)

3.10 How many ways to map \( \text{Sh}_{\mathfrak{gl}_3} \to \text{Sh}_{\mathfrak{gl}_1} \)

First recall that we have 4 specializations \( \text{Sh}_{\mathfrak{gl}_3} \to \text{Sh}_{\mathfrak{gl}_1} \)

Next recall that we have 2 specializations \( \text{Sh}_{\mathfrak{gl}_2} \to \text{Sh}_{\mathfrak{gl}_1} \). Now combining the above two specializations, we have the following possible values of \((\tilde{q}, \tilde{d})\).

\( \tilde{q} = q \) always, and \( \tilde{d} \in \{d^3q^2, d^3, d^3q^{-2}\} \)

3.10.1 Case 1: \( \tilde{q} = q, \tilde{d} = d^3q^2 \)

\[ f (X^k | Y^i | Z^k) \to f (Z^k | q_3Z^k | q_3^2Z^k) \cdot G \]

or

\[ f (X^k | Y^k | Z^k) \to f (Z^k | q_3Z^k | q_3^{-1}Z^k) \cdot G \]

3.10.2 Case 2: \( \tilde{q} = q, \tilde{d} = d^3q^{-2} \)

\[ f (X^k | Y^i | Z^k) \to f (Z^k | q_1^{-1}Z^k | q_1^{-2}Z^k) \cdot G \]

or

\[ f (X^k | Y^k | Z^k) \to f (Z^k | q_1^{-1}Z^k | q_1Z^k) \cdot G \]

3.10.3 Case 3&4: \( \tilde{q} = q, \tilde{d} = d^3 \)

\[ f (X^k | Y^i | Z^k) \to f (Z^k | q_3Z^k | q_3q_1^{-1}Z^k) \cdot G \]

or

\[ f (X^k | Y^k | Z^k) \to f (Z^k | q_3Z^k | q_1Z^k) \cdot G \]

\[ f (X^k | Y^k | Z^k) \to f (Z^k | q_1^{-1}Z^k | q_3q_1^{-1}Z^k) \cdot G \]

\[ f (X^k | Y^k | Z^k) \to f (Z^k | q_1^{-1}Z^k | q_3^{-1}Z^k) \cdot G \]

We obtain the remaining 8 from the above through cyclically rotating, which we demonstrate the notion in the graphs below. Through 2-step method, from \( \text{Sh}_{\mathfrak{gl}_3} \to \text{Sh}_{\mathfrak{gl}_2} \to \text{Sh}_{\mathfrak{gl}_1} \) we are left with 8 specializations that can be represented by the graphs below.

They are:

27
This matches the 12 direct specializations obtained from mapping \( Sh_{gl}^3 \to Sh_{gl}^1 \), which stemmed from three choices of initial point \( i \), and 4 choices of \( q_3 \) vs \( q_1^{-1} \) on edges from \( i \to i + 1 \) and \( i + 1 \to i + 2 \).

Recall that when we tried to generalize from \( Sh_{gl}^3 \to Sh_{gl}^1 \), there were \( 3 \cdot 2^3 - 1 = 12 \) direct specialization maps.

So we expect that all composed specializations \( Sh_{gl}^n \to Sh_{gl}^1 \), there were \( 3 \cdot 2^n - 1 = 12 \) direct specialization maps.

Thus, we have explicitly presented 12 mappings, which is a specific case for the following, more general setting.

3.11 **From** \( Sh_{gl}^{n+1} \) **to** \( Sh_{gl}^1 \)

Here, we have arrived at our main proposition. In order to construct more general homomorphism \( Sh_{gl}^n \to Sh_{gl}^1 \), we next construct the maps of \( \Upsilon_{n+1} : Sh_{gl}^{n+1} \to Sh_{gl}^1 \).
Theorem 3.12 (Second Theorem)

There are \( n \cdot 2^{n-1} \) homomorphisms:

\[
\forall n, \lambda_1, \ldots, \lambda_n \in \{q^{-1}d^{-1}, qd^{-1}\}. \text{ We are interested in the } \lambda_1 = \lambda_2 = \ldots = \lambda_n. \text{ The } G(X^k_l) \text{ should be of the form:}
\]

\[
G(X^k_l) = \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 (x_i - q^2 x_j)^n (x_i - q^{-2} x_j)^n. C_k
\]

For \( f \in \text{Sh}_{\mathbb{Q}^n} (k, \ldots, k) \) and \( g \in \text{Sh}_{\mathbb{Q}^{n+1}} ((l, \ldots, l)) \), the specializations of shuffle factors \( \omega^{(n+1)} \) is:

\[
\prod_{1 \leq i \leq k} \frac{x_i - q^{-2} x_j}{x_i - x_j} \prod_{1 \leq i < j \leq k} \frac{d^{-1} x_i - q \lambda_1 x_j}{x_i - \lambda_1 x_j} \frac{d^{-1} x_i - q \lambda_2 x_j}{x_i - \lambda_2 x_j} \ldots
\]

The ratio of \( G \) is:

\[
\prod_{1 \leq i \leq k} \frac{(x_i - \lambda_1 x_j)(x_i - \lambda_2 x_j) \ldots (x_i - \lambda_n x_j)(x_i - x_j) \prod_{1 \leq r \leq n} \lambda_r^{-1}}{(x_i - x_j)^2 (x_i - q^2 x_j)^n (x_i - q^{-2} x_j)^n}. C_k
\]
Taking the product of the above two factors, we have

\[
\omega^{(n+1)} \left( \frac{X_i^k, \ldots, X_i^k \prod_{i=1}^{n} \lambda_i}{X_{k+1}^l, \ldots, X_{k+1}^l \prod_{i=1}^{n} \lambda_i} \right) \cdot G(X_i^k) = \prod_{1 \leq i \leq k} \left( d^{-1} \prod_{1 \leq r \leq n} \lambda_r \right).
\]

Therefore, we have

\[
\omega^{(n+1)} \left( \frac{X_i^k, \ldots, X_i^k \prod_{i=1}^{n} \lambda_i}{X_{k+1}^l, \ldots, X_{k+1}^l \prod_{i=1}^{n} \lambda_i} \right) \cdot G(X_i^k) = \prod_{1 \leq i \leq k} \left( \omega^{(1)} \left( \frac{x_i}{x_j} \right) \cdot d^{-1} \prod_{1 \leq r \leq n} \lambda_r \right).
\]

\[
\{q_1, q_2, q_3\} = \{q^2, q^{-1}d \prod_{1 \leq r \leq n} \lambda_r^{-1}, q^{-1}d^{-1} \prod_{1 \leq r \leq n} \lambda_r \} \text{ that is } \tilde{q} = q, \tilde{d} = d \prod_{1 \leq r \leq n} \lambda_r^{-1}.
\]

\[
C_k = \prod_{1 \leq i \leq k} (-d^{-1})^{n+1} = (-d)^{-n+1} \tilde{q}_3^{kl}.
\]

We need to verify the wheel conditions for this mapping and choice of \( \lambda \):

1. The \( Sh_{\frac{n-a}{n+1}} \) wheel in colors \((n, n, 0)\) implies vanishing as \( \frac{x_{n-a}}{x_{0,b}} \prod_{1 \leq r \leq n} \lambda_r = qd \) and \( \frac{x_{n-a}}{x_{n,c}} \prod_{1 \leq r \leq n} \lambda_r^{-1} = qd^{-1} \), which means \( \frac{x_{n-a}}{x_{0,b}} = qd \prod_{1 \leq r \leq n} \lambda_r^{-1} = \tilde{q}d = \tilde{q}_3^{-1} \cdot \frac{x_{n,b}}{x_{n,c}} = qd^{-1} \prod_{1 \leq r \leq n} \lambda_r = \tilde{qd}^{-1} = \tilde{q}_3^{-1} \).

2. The \( Sh_{\frac{n-a}{n+1}} \) wheel in colors \((0, 0, n)\) implies vanishing as \( \frac{x_{n,a}}{x_{0,b}} \prod_{1 \leq r \leq n} \lambda_r^{-1} = qd^{-1} \) and \( \frac{x_{n,a}}{x_{n,b}} \prod_{1 \leq r \leq n} \lambda_r = qd \) which means \( \frac{x_{n,a}}{x_{n,b}} = qd^{-1} \prod_{1 \leq r \leq n} \lambda_r^{-1} = \tilde{qd}^{-1} = \tilde{q}_3^{-1} \cdot \frac{x_{n,b}}{x_{n,c}} = qd \prod_{1 \leq r \leq n} \lambda_r^{-1} = \tilde{q}_3^{-1} \).

3. Other cases can be verified similarly.

Therefore, we have constructed a homomorphism \( \Upsilon_{n+1, 1} \).
3.13 From $Sh_{\vartheta_{m+n}}$ to $Sh_{\vartheta_n}$

Finally, we shall establish our main result of Theorem 3.1 by providing all of the details for the claimed construction. Generalizing the above constructions to a more general setting, we map $Sh_{\vartheta_{m+n}} \rightarrow Sh_{\vartheta_n}$. Assuming $n > 2$, we want to construct algebra homomorphism $\Upsilon_{m+n,n}$:

$$\Upsilon_{m+n,n} : \bigoplus_{k_0, k_1, \ldots, k_{n-1}} Sh_{\vartheta_{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1}) |_{q_1, q_2, q_3} \rightarrow Sh_{\vartheta_n} (k_0, \ldots, k_{n-1}) |_{q_1, q_2, q_3}$$

For every $f \left( X^{0,k_0}_{m+1}, X^{1,k_0}_{m+1}, \ldots, X^{m,k_0}_{m+1}, X^{m+1,k_1}_{m+1}, \ldots, X^{m+n+1-1,k_{n-1}}_{m+1} \right) \in Sh_{\vartheta_{m+n}}$, the algebra homomorphism of our Main Theorem is as follows:

**Theorem 3.14 (Main Theorem)** The mapping:

$$\Upsilon_{m+n,n} : \bigoplus_{k_0, k_1, \ldots, k_{n-1}} Sh_{\vartheta_{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1}) |_{q_1, q_2, q_3} \rightarrow Sh_{\vartheta_n} (k_0, \ldots, k_{n-1}) |_{q_1, q_2, q_3}$$
satisfy component-wise homomorphism via:

\[
Y_{m+n,n} : f \left( X_{0,0}^{0,k_0}, X_{1,1}^{1,k_0}, \ldots, X_{m,1}^{m,k_0}, \ldots, X_{m+1,1}^{m+1,k_1}, \ldots, X_{m+n-1,1}^{m+n-1,k_{n-1}} \right) \rightarrow f \left( Z_{0,0}^{0,k_0} \lambda_1 Z_{0,1}^{1,k_0} \ldots, Z_{0,1}^{0,k_0} \mu_1 Z_{1,1}^{1,k_1} \ldots, Z_{n-1,1}^{n-1,k_{n-1}} \prod_{i=1}^{n-1} \mu_i \right) \cdot G
\]

for \( 1 \leq i \leq m, \quad \lambda_i \in \{ q^{-1}d^{-1}, qd^{-1} \} \).

Proof.

We first assume

\[
\lambda_1 = \lambda_2 = \ldots = \lambda_m
\]

3.14.1 Case 1: \( \lambda_1 = \lambda_2 = \ldots = \lambda_m = q^{-1}d^{-1} \)

The rational factor of \( G \left( z_{0,1}^{0,k_0}, \ldots, z_{n-1,1}^{n-1,k_{n-1}} \right) \) should be of the form:

\[
G \left( z_{0,1}^{0,k_0}, \ldots, z_{n-1,1}^{n-1,k_{n-1}} \right) = \frac{\prod_{1 \leq i < j \leq k_0} (z_{0,i} - q^{-1}d^{-1}z_{0,j})^m \prod_{1 \leq j \leq k_0} (q^{-m}d^{-m}z_{0,i} - \mu_1 z_{1,j})}{\prod_{1 \leq j \leq k_1} (z_{1,i} - q^2z_{1,j}) \prod_{1 \leq i \leq k_1} (z_{1,i} - z_{2,j}) \ldots \prod_{1 \leq j \leq k_{n-1}} (z_{n-1,i} - z_{0,j})} \cdot \mu_r \]

For \( f \in Sh_{q^{k_0}m} \left( k_0, \ldots, k_0, k_1, \ldots, k_{n-1} \right) \) and \( g \in Sh_{q^{k_0}m} \left( l_0, \ldots, l_0, l_1, \ldots, l_{n-1} \right) \), that is

\[
f \left( Z_{0,0}^{0,k_0}, \frac{1}{q^{k_0}} Z_{0,0}^{0,k_0}, \ldots, \frac{1}{q^{m}d^{m}} Z_{0,0}^{0,k_0}, \mu_1 Z_{1,1}^{1,k_1}, \ldots, Z_{n-1,1}^{n-1,k_{n-1}} \prod_{r=1}^{n-1} \mu_r \right)
\]

\[
g \left( Z_{0,0}^{0,k_0}, \frac{1}{q^{k_0}} Z_{0,0}^{0,k_0}, \ldots, \frac{1}{q^{m}d^{m}} Z_{0,0}^{0,k_0}, \mu_1 Z_{1,1}^{1,k_1}, \ldots, Z_{n-1,1}^{n-1,k_{n-1}} \prod_{r=1}^{n-1} \mu_r \right)
\]

32
The specialization of $\omega^{(m+n)} \left( \frac{z_{0,k_0} \cdots z_{n-1,k_{n-1}}}{\mu_r} \prod_{r=1}^{n-1} \mu_r \right)$ is:

$$
\begin{align*}
&\prod_{1 \leq i \leq k_0} \left( \frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}} \right)^{m+1} \left( \frac{d_{-1} z_{0,i} - d_{-1} z_{0,j}}{z_{0,i} - q^{-1}z_{0,j}} \right)(z_{0,i} - q^2 z_{0,j})^m \left( \frac{z_{0,i} - z_{0,j}}{z_{0,i} - q dz_{0,j}} \right), \\
&\prod_{1 \leq i \leq k_1} \mu_1 z_{1,i} - q^{-m-1} \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - q^{-m-1} d_{-1} z_{0,j}}{z_{0,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{z_{n-1,i} - d_{-1} \mu_r - q z_{0,j} a_{n-1} \leq i \leq k_{n-1} + l_{n-1}}{z_{n-1,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{z_{n-1,i} - q^{-m} d_{-1} z_{0,i} - q \mu_1 z_{1,j}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{z_{n-1,i} - d_{-1} \mu_r - q \mu_1 z_{1,j}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{n-1 \mu_r - q z_{0,j} a_{n-1} \leq i \leq k_{n-1} + l_{n-1}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{n-1 \mu_r - q \mu_1 z_{1,j}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
\end{align*}
$$

Pulling out the constants of (21), the ratio of $G$ is:

$$
\begin{align*}
&\prod_{1 \leq i \leq k_0} \left( \frac{z_{0,i} - q^{-1}d_{-1} z_{0,j}}{z_{0,i} - q^{-1}d_{-1} z_{0,j}} \right)^m \left( \frac{z_{0,j} - q^{-1}d_{-1} z_{0,i}}{z_{0,j} - q^{-2}z_{0,j}} \right)^m. \\
&\prod_{1 \leq i \leq k_1} \frac{q^{-m} d_{-1} z_{0,i} - \mu_1 z_{1,j}}{z_{0,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_0} \frac{z_{n-1,i} - \mu_r - q z_{0,j} a_{n-1} \leq i \leq k_{n-1} + l_{n-1}}{z_{n-1,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_0} \frac{z_{n-1,i} - \mu_1 z_{1,j}}{z_{n-1,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{z_{n-1,i} - d_{-1} \mu_r - q z_{0,j} a_{n-1} \leq i \leq k_{n-1} + l_{n-1}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{z_{n-1,i} - \mu_1 z_{1,j}}{z_{n-1,i} - z_{0,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{n-1 \mu_r - q \mu_1 z_{1,j}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
&\prod_{1 \leq i \leq k_{a-1}} \frac{n-1 \mu_r - q \mu_1 z_{1,j}}{z_{n-1,i} - \mu_1 z_{1,j}}. \\
\end{align*}
$$

Calculating the product of (22) and (23) involves massive calculations. But we can determine $	ilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}, \tilde{d}$ and $G$ by analyzing $\rho(z_{u,v}, z_{s,t})$ which is the product of terms containing $(z_{u,v}, z_{s,t})$, $0 \leq u, s \leq n-1$, $1 \leq v, t \leq \max (k_r + l_r)$, $1 \leq r \leq n-1$. 

33
1. $\rho(z_{0,i}, z_{0,j})$ in the product is:

$$
\rho(z_{0,i}, z_{0,j}) = \prod_{1 \leq i \leq k_0} \left( \frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}} \right)^{m+1} \left( \frac{d^{-1}z_{0,i} - d^{-1}z_{0,j}}{z_{0,i} - z_{0,j}} \right)^{m}.
$$

$$
\rho(z_{0,i}, z_{0,j}) = \left( -q^{-1}d^{-2} \right)^{k_0} \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}}.
$$

We know $\frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}}$ is the shuffle factors of $\omega_{i,i}^{(n)} \left( \frac{z_{0,i}}{z_{0,j}} \right)$, when $q = q$.

2. $\rho(z_{0,i}, z_{1,j})$ with $1 \leq i \leq k_0$, $k_1 < j \leq k_1 + l_1$ in the product is:

$$
\rho(z_{0,i}, z_{1,j}) = \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - q^{m+1}d_{m+1} + 1 \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \frac{z_{0,i} - q^{m+1}d_{m+1} + 1 \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \frac{1}{q^{m}d^{m+1}}.
$$

In order to identify $\rho(z_{0,i}, z_{1,j})$ with const $\cdot \prod_{1 \leq i \leq k_0} \omega_{i,i}^{(n)} \left( \frac{z_{0,i}}{z_{1,j}} \right)$, we must have $\tilde{d} = q^{m+1}d_{m+1} + 1 \mu_1$.

3. $\rho(z_{1,i}, z_{0,j})$ with $1 \leq i \leq k_1$, $k_0 < j \leq k_0 + l_0$ in the product is:

$$
\rho(z_{1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_1} \frac{z_{1,i} - q^{-(m-1)}d^{-(m+1)} + 1 \mu_1 z_{0,j}}{z_{1,i} - z_{0,j}}.
$$

$$
\rho(z_{1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_1} \left( \frac{z_{1,i} - q^{-(m-1)}d^{-(m+1)} + 1 \mu_1 z_{0,j}}{z_{1,i} - z_{0,j}} \right) \cdot \mu_1.
$$

In order to identify $\rho(z_{1,i}, z_{0,j})$ with const $\cdot \prod_{1 \leq i \leq k_1} \omega_{i,i}^{(n)} \left( \frac{z_{1,i}}{z_{0,j}} \right)$, we must have $\tilde{d} = q^{m}d^{m+1} + 1 \mu_1$, $q_{1,i} = q^{-1}d^{-(m+1)} + 1 \mu_1$, and $\tilde{d}^{-1} = q^{-(m-1)}d^{-(m+1)} + 1 \mu_1^{-1}$, we have $d = q^{m}d^{m+1} + 1 \mu_1$, as $q = \tilde{q}$.

4. $\rho(z_{0,i}, z_{n-1,j})$ with $1 \leq i \leq k_0$, $k_{n-1} < j \leq k_{n-1} + l_{n-1}$ in the product is:
5. \( \rho(z_{n-1,i}, z_{0,j}) \) with \( 1 \leq i \leq k_{n-1}, k_0 < j \leq k_0 + l_0 \) in the product is:

\[
\rho(z_{n-1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_{n-1}} \frac{z_{n-1,i} - q d z_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1} \cdot z_{n-1,i} - z_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1}}{z_{n-1,i} - z_{0,j}}.
\]

In order to identify \( \rho(z_{0,i}, z_{n-1,j}) \) with \( k_{n-1} \leq j \leq k_{n-1} + l_0 \),
we must have \( \tilde{q} d^{-1} = q d^{-1} \prod_{r=1}^{n-1} \mu_r^{-1} \).

6. \( \rho(z_{a,i}, z_{0,j}) \) with \( 1 \leq i \leq k_a, k_a < j \leq k_a + l_a \) in the product is:

\[
\rho(z_{a,i}, z_{0,j}) = \prod_{1 \leq i \leq k_a} \frac{z_{a,i} - q^{-2} z_{0,j} \prod_{r=1}^{l_a} \omega^{(n)}_{i,i+1} \left( \frac{z_{a,i}}{z_{a,j}} \right)}{z_{a,i} - z_{0,j} \prod_{r=1}^{l_a} \omega^{(n)}_{i,i+1} \left( \frac{z_{a,i}}{z_{a,j}} \right)}.
\]

given \( \tilde{q} = q \).

7. \( \rho(z_{a,i}, z_{q+1,j}) \) with \( 1 \leq i \leq k_a, k_a + 1 < j \leq k_a + 1 + l_{a+1}, 1 \leq a \leq n - 2 \) in the product is:
In order to identify $\rho(z_{a,i}, z_{a+1,j})$ with $1 \leq i \leq k_a, k_a < j \leq k_a + l_a, 1 \leq a \leq n - 2$ in the product is:

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_a+1} \frac{z_{a+1,i} - q d \mu_{a+1} z_{a+1,j}}{z_{a+1,i} - \mu_{a+1} z_{a+1,j}} \cdot \frac{1}{d}, \quad z_{a,i} = \mu_{a+1} z_{a+1,j}$$

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_a+1} \left( \frac{z_{a+1,i} - q d \mu_{a+1} z_{a+1,j}}{z_{a+1,i} - z_{a,j}} \cdot \frac{1}{d} \right)$$

In order to identify $\rho(z_{a,i}, z_{a+1,j})$ with $k_{a+1} < j < k_{a+1} + l_{a+1}$, we must have $\tilde{q} d = q d \mu_{a+1}$ which means $\tilde{q} = q$ and $\tilde{d} = d \mu_{a+1}$.

8. $\rho(z_{a+1,i}, z_{a,j})$ with $1 \leq i \leq k_a+1, k_a < j \leq k_a + l_a, 1 \leq a \leq n - 2$ in the product is:

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_a+1} \frac{z_{a+1,i} - q d^{-1} \mu_{a+1} z_{a,j}}{z_{a+1,i} - \mu_{a+1} z_{a,j}} \cdot \mu_{a+1}$$

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_a+1} \left( \frac{z_{a+1,i} - q d^{-1} \mu_{a+1} z_{a,j}}{z_{a+1,i} - z_{a,j}} \cdot \mu_{a+1} \right)$$

In order to identify $\rho(z_{a+1,i}, z_{a,j})$ with $k_{a+1} < j < k_{a+1} + l_{a+1}$, we must have $\tilde{q} d^{-1} = q d^{-1} \mu_{a+1}$ which is guaranteed by $\rho(z_{a,i}, z_{a+1,j})$.

To summarize, we have obtained that:

$$\tilde{d} = q^m d^{m+1} \mu_1, \quad \tilde{d} = d \mu_2, \ldots, \tilde{d} = d \mu_{n-1}, \quad \tilde{d} = d \prod_{r=1}^{n-1} \mu_r^{-1}$$

Taking the product, we have: $\tilde{d}^n = q^m d^{m+n}$ and $\tilde{d} = q^m d^{m+1} = d q_1^{\frac{m}{n}}$.

Since $q_1 = d q^{-1}, q_3 = d^{-1} q^{-1}$, we have $d = q_1^{-\frac{1}{q_3}} q_3^{-\frac{1}{q_3}}$. Therefore,

$$q_1 = q_1 q_3^{\frac{m}{n}}, \quad q_2 = q_2, \quad q_3 = q_3 q_3^{\frac{m}{n}}$$

$$\mu_2 = \mu_3 = \ldots = \mu_{n-1} = d d^{-1} = q^m d^m = q_3^{\frac{m}{n}}$$

$$\mu_1 = d q^{-m} d^{-(m+1)} = (qd)^{\frac{m}{n}} = q_3^{\frac{m}{n}}$$

Now, let’s determine the constants $C_k$. We know that

$$\frac{C_{k+1}}{C_k C_1} \left( -\frac{1}{q d^2} \right) \frac{k_{a_1}}{k_{a_2}} \mu_1 k_{a_1} \mu_2 k_{a_2} \ldots \mu_{n-1} k_{a_{n-2}} \left( \frac{\tilde{d}}{d} \right)^{\frac{n-2}{a_1}} \sum_{a=1}^{k_{a_1}} k_{a_1} a_1 = 1$$

$$\frac{\tilde{d}}{d} q^m d^{m+1} = \mu_1, \quad \text{and} \quad \mu_2 = \ldots = \mu_{n-1} = \frac{\tilde{d}}{d}$$

36
Therefore,
\[ \frac{C_{k+1}}{C_k C_l} \prod_{a=1}^{n-1} \mu_a^{-k_{a+1}} \cdot (-\frac{1}{qd^2})^{k_0} = 1 \]

In particular, we can choose \( C_k = \prod_{a=1}^{n-1} \mu_a^{-k_{a+1}} \cdot (-qd^2)^{k_0} \).

It remains to verify wheel conditions:

1. The \( Sh_{\theta^{m+n}} \) wheel in color \((r, r, r + 1) 1 \leq r \leq n - 2 \) implies vanishing as
   \( \frac{\text{q}_3}{r+1} \cdot \frac{\text{z}_{r, a}}{\text{z}_{r+1, b}} = qd, \frac{\text{q}_3}{r+1} \cdot \frac{\text{z}_{r+1, a}}{\text{z}_{r, c}} = qd^{-1} \), which means \( \frac{\text{z}_{r, a}}{\text{z}_{r+1, b}} = \hat{q} d, \frac{\text{z}_{r+1, a}}{\text{z}_{r, c}} = \hat{q} d^{-1} \).

2. The \( Sh_{\theta^{m+n}} \) wheel in color \((0, 0, 1) \) and \((n - 1, n - 1, 0) \) are verified as above by using \( \hat{q} d^{-1} = \prod_{1 \leq r \leq n-1} \mu_r^{-1} = q_3^{-n}, \mu_1 q_3^{-n} = q_3^{-n} \).

3. The \( Sh_{\theta^{m+n}} \) wheel in color \((r + 1, r + 1) \) implies vanishing as \( q_3^{-n} \).
   \( \frac{\text{z}_{r+1, a}}{\text{z}_{r, b}} = qd^{-1}, \text{q}_3^{-n} \cdot \frac{\text{z}_{r+1, b}}{\text{z}_{r, c}} = qd, \) which means \( \frac{\text{z}_{r, a}}{\text{z}_{r+1, b}} = \hat{q} d, \frac{\text{z}_{r+1, a}}{\text{z}_{r, c}} = \hat{q} d^{-1} \).

4. The \( Sh_{\theta^{m+n}} \) wheel in color \((n - 1, 0, 0) \) and \((0, 1, 0) \) are verified in the same way.

Therefore, we have constructed a homomorphism \( Y_{m+n,n} \)

\[
Y_{m+n,n} : \bigoplus_{k_0, k_1, \ldots, k_{n-1}} Sh_{\theta^{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1})_{q_1, q_2, q_3} \to
\]

given by

\[
Y_{m+n,n} : f \left( X_{0,0}^{m,k_0}, X_{1,0}^{m,k_0}, \ldots, X_{m,0}^{m,k_0}, X_{0,1}^{m+1,k_1}, \ldots, X_{m,1}^{m+1,k_1}, \ldots, X_{m+n-1,0}^{m+n-1,k_{n-1}} \right) \to
\]

\[
f \left( z_0^{k_0}, z_3^{k_0}, \ldots, q_{a}^{k_0}, z_3^{k_0}, \ldots, z_3^{k_0}, \ldots, z_3^{k_0}, \ldots, q_{a}^{k_0}, \ldots, z_3^{k_0}, \ldots, z_3^{k_0}, \ldots, z_3^{k_0} \right) \cdot G
\]

\[
G = \frac{\prod_{1 \leq i,j \leq k_0} (z_{0,i} - q^{-1}d^{-1}z_{0,j})^{m} \cdot \prod_{a=0}^{n-1} \prod_{1 \leq i,j \leq k_a} (z_{a,i} - q^{a-1}z_{a+1,j})^{m}}{\prod_{1 \leq i,j \leq k_0} (z_{0,i} - q^{2}z_{0,j})^{m}, (z_{0,i} - q^{-2}z_{0,j})^{m}} \cdot \frac{\prod_{a=0}^{n-1} \prod_{1 \leq i,j \leq k_a} (z_{a,i} - z_{a+1,j})^{m}}{\prod_{a=0}^{n-1} \prod_{1 \leq i,j \leq k_a} (z_{a,i} - z_{a+1,j})^{m}}
\]

37
3.14.2 Case 2: $\lambda_1 = \lambda_2 = \ldots = \lambda_m = qd^{-1}$

The rational factor of $G\left(z_{0,0}^{n,0}, \ldots, z_{n-1,n-1}^{n-1,k_{n-1}}\right)$ should be of the form:

$$G\left(z_{0,0}^{n,0}, \ldots, z_{n-1,n-1}^{n-1,k_{n-1}}\right) =$$

$$\prod_{1 \leq i \leq k_0} (z_{0,i} - qd^{-1}z_{0,j})^{m_{i,j}} \cdot \prod_{1 \leq i \leq k_0} (q^m d^{-m}z_{0,i} - \mu_1 z_{1,j}) \cdot \prod_{1 \leq j \leq k_1} (z_{0,i} - z_{1,j}) \cdot \prod_{1 \leq j \leq k_2} (z_{1,i} - z_{2,j}) \cdot \prod_{1 \leq i \leq k_{n-1}} (z_{n-1,i} - z_{0,j}) \cdot \left( \prod_{1 \leq j \leq k_n} (z_{n-1,i} - z_{0,j}) \right) \cdot C_k$$

For $f \in \mathcal{Sh}_{\frak{g}_{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1})$ and $g \in \mathcal{Sh}_{\frak{g}_{m+n}} (l_0, \ldots, l_0, l_1, \ldots, l_{n-1})$.

That is

$$f \left( Z_{0,0}^0, \ldots, Z_{0,0}^m, \ldots, Z_{0,1}^1, \ldots, Z_{n-1,1}^1, \ldots, Z_{n-1,n-1}^{n-1,k_{n-1}} \prod_{r=1}^{n-1} \mu_r \right)$$

$$g \left( Z_{0,0}^{k_0+h_0}, \ldots, Z_{0,0}^{m+k_0+h_0}, \ldots, Z_{0,1}^{k_1+k_1+1}, \ldots, Z_{0,1}^{n+k_1+1}, \ldots, Z_{n-1,n-1}^{n-1,k_{n-1}+l_{n-1}} \prod_{r=1}^{n-1} \mu_r \right)$$

The ratio of factor of $G$ ignoring the constants, is:

$$\frac{1}{1 \leq i \leq k_0} \left( z_{0,i} - qd^{-1}z_{0,j} \right)^{m_{i,j}} \cdot \frac{1}{1 \leq j \leq k_1} \left( z_{0,i} - z_{1,j} \right)^{m_{i,j}} \cdot \frac{1}{1 \leq k_0 \leq h_0} \left( z_{0,i} - qd^{-1}z_{0,j} \right)^{m_{i,j}}$$

$$\frac{1}{1 \leq j \leq k_2} \left( z_{1,i} - z_{2,j} \right)^{m_{i,j}} \cdot \frac{1}{1 \leq i \leq k_{n-1}} \left( z_{n-1,i} - z_{0,j} \right)^{m_{i,j}}$$

$$\frac{1}{1 \leq j \leq k_n} \left( z_{n-1,i} - z_{0,j} \right)^{m_{i,j}}$$

(24)

The specialization of $\omega^{(n)} \left( \prod_{1 \leq j \leq k_0} z_{0,i}^{m_{i,j}} \prod_{1 \leq j \leq k_1} z_{1,i}^{m_{i,j}} \prod_{1 \leq j \leq k_2} z_{2,j}^{m_{i,j}} \prod_{1 \leq j \leq k_{n-1} \leq k_{n-1} + l_{n-1}} z_{n-1,i}^{m_{i,j}} \prod_{1 \leq j \leq k_n} z_{n,i}^{m_{i,j}} \prod_{1 \leq a \leq n-1} \mu_a \prod_{1 \leq i \leq k_a} z_{a,i}^{m_{i,j}} \prod_{1 \leq a \leq n-2} \mu_a \prod_{1 \leq i \leq k_{a+1}} z_{a+1,i}^{m_{i,j}} \right)$ is:

38
We can determine $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}, \tilde{d}$ and $G$ using the same method as above cases by analysing the product of terms in (25) and (24):

1. $\rho(z_{0,i}, z_{0,j})$ in the product is:

$$
\rho(z_{0,i}, z_{0,j}) = \prod_{1 \leq i \leq k_0} \left( \frac{z_{0,i} - q^{-2} z_{0,j}}{z_{0,i} - z_{0,j}} \right)^{m+1} \cdot \left( \frac{z_{0,i} - q^2 z_{0,j}}{z_{0,i} - q d^{-1} z_{0,j}} \right)^m \cdot \left( \frac{z_{0,i} - q d^{-1} z_{0,j}}{z_{0,i} - q d^{-1} z_{0,j}} \right)^m.
$$

$$
\rho(z_{0,i}, z_{0,j}) = (-qd^{-2})^{k_0} \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - q^{-2} z_{0,j}}{z_{0,i} - z_{0,j}}.
$$

If $\tilde{q} = q$ then

$$
\rho(z_{0,i}, z_{0,j}) = (-qd^{-2})^{k_0} \prod_{1 \leq i \leq k_0} \omega_1, i \left( \frac{z_{0,i}}{z_{0,j}} \right).
$$

2. $\rho(z_{0,i}, z_{1,j})$ with $1 \leq i \leq k_0, k_1 \leq j \leq k_1 + l_1$ in the product is:

$$
\rho(z_{0,i}, z_{1,j}) = \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - q^{-(m-1)} d^{m+1} \mu_1 z_{1,j}}{z_{0,i} - q^{-m} d^m \mu_1 z_{1,j}} \cdot \frac{z_{0,i} - q^m d^{-m} \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \frac{q^m d^{-m}}{d^{m+1}}.
$$

$$
\rho(z_{0,i}, z_{1,j}) = \prod_{1 \leq i \leq k_0} \left( \frac{z_{0,i} - q^{-(m-1)} d^{m+1} \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \right) \cdot \frac{q^m}{d^{m+1}}.
$$
In order to identify $\rho(z_{0,i}, z_{1,j})$ with \( const \cdot \prod_{1 \leq i \leq k_0} \omega_{i,i+1}^{(n)} \left( \frac{z_{0,i}}{z_{1,i}} \right) \), we must have \( \tilde{q}d = q^{-(m-1)}d^{m+1}\mu_1 \). That is \( d = q^{-m}d^{m+1}\mu_1 \).

3. $\rho(z_{1,i}, z_{0,j})$ with $1 \leq i \leq k_1, k_0 < j \leq k_0 + l_0$ in the product is:

\[
\rho(z_{1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_1} \frac{z_{1,i} - q^{m+1}d^{-(m+1)}\mu_1^{-1}z_{0,j}}{z_{1,i} - q^{m}d^{-m}\mu_1^{-1}z_{0,j}}, \quad \frac{z_{1,i} - q^{m}d^{-m}\mu_1^{-1}z_{0,j}}{z_{1,i} - z_{0,j}} \cdot \mu_1
\]

In order to identify $\rho(z_{1,i}, z_{0,j})$ with \( const \cdot \prod_{1 \leq i \leq k_1} \omega_{i,i+1}^{(n)} \left( \frac{z_{1,i}}{z_{0,j}} \right) \), we must have \( \tilde{q}d^{-1} = q^{m+1}d^{-(m+1)}\mu_1^{-1} \).

4. $\rho(z_{0,i}, z_{n-1,j})$ with $1 \leq i \leq k_0, k_{n-1} < j \leq k_{n-1} + l_{n-1}$ in the product is:

\[
\rho(z_{0,i}, z_{n-1,j}) = \prod_{1 \leq i \leq k_0} \frac{z_{0,i} - qd^{-1}z_{n-1,j} \prod_{r=1}^{n-1} \mu_r}{z_{0,i} - z_{n-1,j} \prod_{r=1}^{n-1} \mu_r}, \quad \frac{z_{0,i} - z_{n-1,j} \prod_{r=1}^{n-1} \mu_r}{z_{0,i} - z_{n-1,j}}
\]

In order to identify $\rho(z_{0,i}, z_{n-1,j})$ with \( \prod_{1 \leq i \leq k_0} \omega_{i,i+1}^{(n)} \left( \frac{z_{0,i}}{z_{n-1,j}} \right) \), we must have \( \tilde{q}d^{-1} = qd^{-1} \prod_{r=1}^{n-1} \mu_r \), which means \( d = d \prod_{r=1}^{n-1} \mu_r^{-1} \).

5. $\rho(z_{n-1,i}, z_{0,j})$ with $1 \leq i \leq k_{n-1}, k_0 < j \leq k_0 + l_0$ in the product is:

\[
\rho(z_{n-1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_{n-1}} \frac{z_{n-1,i} - qdz_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1}}{z_{n-1,i} - z_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1}}, \quad \frac{z_{n-1,i} - z_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot \frac{d^{-1} \prod_{r=1}^{n-1} \mu_r}{d^{-1} \prod_{r=1}^{n-1} \mu_r}
\]

\[
\rho(z_{n-1,i}, z_{0,j}) = \prod_{1 \leq i \leq k_{n-1}} \left( \frac{z_{n-1,i} - qdz_{0,j} \prod_{r=1}^{n-1} \mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot \frac{d^{-1} \prod_{r=1}^{n-1} \mu_r}{d^{-1} \prod_{r=1}^{n-1} \mu_r} \right)
\]
In order to identify \( \rho(z_{n-1,i}, z_{0,j}) \) with \( \text{const} \cdot \prod_{1 \leq i \leq k_{a-1}} \omega_{i,i+1}^{(n)} \left( \frac{z_{n-1,i}}{z_{0,j}} \right) \), we must have \( \tilde{q} \tilde{d} = qd \prod_{r=1}^{n-1} \mu_{r}^{-1} \), which is guaranteed by \( \rho(z_{0,i}, z_{a-1,j}) \).

6. \( \rho(z_{a,i}, z_{a,j}) \) with \( 1 \leq i \leq k_{a} \), \( k_{a} < j \leq k_{a} + l_{a} \) in the product is:

\[
\rho(z_{a,i}, z_{a,j}) = \prod_{1 \leq i \leq k_{a}} \frac{z_{a,i} - q^{-2} z_{a,j}}{z_{a,i} - z_{a,j}} = \prod_{1 \leq i \leq k_{a}} \omega_{i,i}^{(n)} \left( \frac{z_{a,i}}{z_{a,j}} \right)
\]

We have \( \tilde{q} = q \).

7. \( \rho(z_{a,i}, z_{a+1,j}) \) with \( 1 \leq i \leq k_{a} \), \( k_{a} < j \leq k_{a} + l_{a} + 1 \) \( \forall 1 \leq a \leq n - 2 \) in the product is:

\[
\rho(z_{a,i}, z_{a+1,j}) = \prod_{1 \leq i \leq k_{a}} \frac{z_{a,i} - qd \mu_{a+1} z_{a+1,j}}{z_{a,i} - \mu_{a+1} z_{a+1,j}} \cdot \frac{1}{d} \frac{z_{a,i} - \mu_{a+1} z_{a+1,j}}{z_{a,i} - z_{a+1,j}}
\]

\[
\rho(z_{a,i}, z_{a+1,j}) = \prod_{1 \leq i \leq k_{a}} \left( \frac{z_{a,i} - qd \mu_{a+1} z_{a+1,j}}{z_{a,i} - z_{a+1,j}} \right) \cdot \frac{1}{d}
\]

In order to identify \( \rho(z_{a,i}, z_{a+1,j}) \) with \( \prod_{1 \leq i \leq k_{a}} \omega_{i,i+1}^{(n)} \left( \frac{z_{a,i}}{z_{a+1,j}} \right) \), we must have \( \tilde{q} \tilde{d} = qd \mu_{a+1} \) which means \( \tilde{d} = d \mu_{a+1} \).

8. \( \rho(z_{a+1,i}, z_{a,j}) \) with \( 1 \leq i \leq k_{a+1}, k_{a} < j \leq k_{a} + l_{a} \) \( \forall 1 \leq a \leq n - 2 \) in the product is:

\[
\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_{a+1}} \frac{z_{a+1,i} - qd^{-1} \mu_{a}^{-1} z_{a,j}}{z_{a+1,i} - \mu_{a+1} z_{a,j}} \cdot \mu_{a+1}
\]

\[
\rho(z_{a+1,i}, z_{a,j}) = \prod_{1 \leq i \leq k_{a+1}} \left( \frac{z_{a+1,i} - qd^{-1} \mu_{a}^{-1} z_{a,j}}{z_{a+1,i} - z_{a,j}} \right) \cdot \mu_{a+1}
\]

In order to identify \( \rho(z_{a+1,i}, z_{a,j}) \) with \( \prod_{1 \leq i \leq k_{a+1}} \omega_{i,i+1}^{(n)} \left( \frac{z_{a+1,i}}{z_{a,j}} \right) \), we must have \( \tilde{q} \tilde{d}^{-1} = qd^{-1} \mu_{a+1}^{-1} \) which is guaranteed by \( \rho(z_{a,i}, z_{a+1,j}) \).

It remains to verify wheel conditions:

1. The \( S_{h} \) wheel in color \( (r, r + 1) \) \( 1 \leq r \leq n - 2 \) implies vanishing as \( q_{1}^{m} \cdot \frac{z_{r,a}}{z_{r+1,b}} = qd \cdot q_{1}^{\frac{m}{2}} \cdot \frac{z_{r+1,b}}{z_{r,c}} = qd^{-1} \), which means \( \frac{z_{r,a}}{z_{r+1,b}} = qd \cdot q_{1}^{\frac{m}{2}} = qd^{-1} \).
2. The $\text{Sh}_{\Theta_{m+n}}$ wheel in color $(r, r + 1, r + 1)$ implies vanishing as $q_1^{m+1} \cdot \frac{\overline{z}_{r+1,c}}{z_{r,b}} = \overline{q}d_{r-1} q_1^{m} = qd$, which means $\frac{\overline{z}_{r,b}}{z_{r+1,c}} = \overline{q}d_{r-1}$. 

3. Other cases can be verified similarly.

Therefore, we have constructed a homomorphism $\Upsilon_{m+n,n}$.

\[
\Upsilon_{m+n,n} : \bigoplus_{k_0, k_1, \ldots, k_{n-1}} \text{Sh}_{\Theta_{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1}) |_{q_1, q_2, q_3} \rightarrow \text{Sh}_{\Theta_{m+n}} (k_0, \ldots, k_0, k_1, \ldots, k_{n-1}) |_{q_1, q_2, q_3} \quad \text{given by}
\]

\[
\Upsilon_{m+n,n} : f (X^{0, k_0}_{0,1}, X^{1, k_0}_{1,1}, \ldots, X^{m, k_0}_{m,1}, \ldots, X^{m+n-1, k_0}_{m+n-1,1}) \rightarrow f \left( Z^{0, k_0}_{0,1}, q_1 z^{0, k_0}_{0,1}, \ldots, q_1^{m+n-1} Z^{1, k_1}_{1,1}, \ldots, q_1^{m+n-1} Z^{n-1, k_{n-1}}_{m+n-1,1} \right) \cdot G
\]

\[
G = \prod_{1 \leq i, j \leq k_0} \left( z_{0,i} - q^{d_{r-1}} z_{0,j} \right)^m \cdot \prod_{a=0}^{n-1} \prod_{1 \leq i \leq k_0} \left( z_{a,i} - q^{d_{r-1}} z_{a+1,j} \right)^m \cdot \prod_{a=1}^{n-1} \mu_a^{-k_a} \cdot (-qd)^{k_a^2} \cdot \prod_{a=0}^{n-1} \prod_{1 \leq i \leq k_0} \left( z_{a,i} - z_{a+1,j} \right).
\]

To summarize, we have obtained that:

\[
d = q^{-m} d^{m+1} \mu_1, \quad \bar{d} = d \mu_2, \ldots, \bar{d} = d \mu_{n-1}, \quad \overline{d} = d \prod_{r=1}^{n-1} \mu_r^{-1}
\]

Taking the product of the terms above, we have: $\overline{d}^n = q^{-m} d^{m+n} = d(qd)^{-1} d^{m+1}$ and $\overline{d} = d(qd)^{-1} \overline{d} = dq_1^{m}$. 

In particular,

\[
q_1 = q_1 q_1^{m}, \quad q_2 = q_3 q_1^{m}, \quad \mu_2 = \mu_3 = \ldots = \mu_{n-1} = \overline{d}^{-1} = q_1^{m},
\]

\[
\mu_1 = d \overline{d}^{m} d^{-(m+1)} = q_1^{-m+1}
\]

Thus, overall we get $2^{n-1}$ such specialization homomorphisms when we start from color 0 variables. Allowing the extra rotation, we end up having precisely $n \cdot 2^{n-1}$ specialization homomorphisms.
4 Compatibility of subalgebra mappings

We conclude our paper with a few compatibility properties among realization maps and important subalgebras previously studied in [FHHSY], [FT]. Under the constructed homomorphism $\bigoplus_{\mathfrak{g} = (k_0, \ldots, k_m)} Sh_{\mathfrak{g}^{m+n}} \rightarrow Sh_{\mathfrak{g}^{m+n}}$, we explore the image and behavior of subalgebras under this mapping.

4.1 Bethe Subalgebras

Define Bethe subalgebra as a $\mathbb{N}$-graded subspace. Following [FHHSY] and Definition 1.8 of [FT], we introduce an important $\mathbb{Z}_+$-graded subspace $A^{sm} = \bigoplus_n A_n^{sm}$ of $Sh_{\mathfrak{g}^{m+n}}$. Its degree $n$ component is defined by

$$A_n^{sm} := \left\{ F \in Sh_{\mathfrak{g}^{m+n}} \mid \partial^{(0,k)} F, \partial^{(\infty,k)} F \text{ exist and } \partial^{(0,k)} F = \partial^{(\infty,k)} F \quad 0 \leq k \leq n \right\}$$

where $\partial^{(0,k)} F := \lim_{\xi \to 0} F(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n)$

$\partial^{(\infty,k)} F := \lim_{\xi \to \infty} F(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n)$

With the particular specialization discussed above, we have:

Theorem 4.2 (Mapping Compatibility with Bethe Subalgebras)

$$\mathcal{T}_{m+n,n} : f \left( X_{0,N}^0, X_{1,N}^1, \ldots, X_{m+N}^m, X_{m+1,N}^{m+1}, \ldots, X_{m+n-1,N}^{m+n-1} \right) \rightarrow$$

$$F \left( X_{0,N}^0, q_1 X_{0,N}^0, \ldots, q_m X_{0,N}^m, q_m Z_{1,1}^m, \ldots, q_m Z_{n-1,1}^m \right) \cdot G$$

$$G = \frac{\text{const} \cdot \prod_{i,j=1}^N \left( z_{0,i} - q_3 z_{0,j} \right) m \prod_{a=0}^{n-1} \prod_{i,j=1}^N \left( z_{a,i} - q_3 z_{a+1,j} \right) \prod_{a=0}^{n-1} \prod_{i,j=1}^N \left( z_{a,i} - z_{a+1,j} \right) \prod_{i,j=1}^N \left( z_{0,i} - q^2 z_{0,j} \right) m \prod_{a=0}^{n-1} \prod_{i,j=1}^N \left( z_{0,i} - q^{-2} z_{0,j} \right) m}{\prod_{1 \leq i < j \leq N} \left( (z_{0,i} - q^2 z_{0,j}) (z_{0,i} - q^{-2} z_{0,j}) \right) m}$$

and the new parameters are $\tilde{q}_1 = q_1 q_3^{-m}, \tilde{q}_2 = q_2, \tilde{q}_3 = q_3 q_3^{-m}$.

Proof.

Note that any segment $[a', b']$ rolled around $\mathbb{Z}/n\mathbb{Z}$ is arising from a segment $[a, b]$ rolled around $\mathbb{Z}/m + n\mathbb{Z}$.

Also recall the explicit generators of $A(\mathfrak{s})$:

$$F_N^n (\mathfrak{s}) = \prod_{i=0}^{m+n-1} \prod_{r \neq s \leq N} \left( x_{i,r} - q^{-2} x_{i,s} \right) \prod_{i=0}^{m+n-1} \left( s_0 \ldots s_i \prod_{r=1}^{N} x_{i,r} - \mu \cdot \prod_{r=1}^{N} x_{i+1,r} \right) \prod_{i=0}^{m+n-1} \prod_{r,s=1}^{N} \left( x_{i,r} - x_{i+1,s} \right)$$

43
Note that under above specialization:

\[
\prod_{r=0}^{m+n-1} \prod_{r \neq s \leq N} (x_{i,r} - q^{-2}x_{i,s}) \rightarrow \prod_{i=0}^{n-1} \prod_{r \neq s \leq N} (z_{i,r} - q^{-2}z_{i,s})
\]

\[
\prod_{i=0}^{m+n-1} \prod_{r,s=1}^{N} (x_{i,r} - x_{i+1,s}) \rightarrow \prod_{a=0}^{n-1} \prod_{r,s=1}^{N} (z_{a,r} - z_{a+1,s})
\]

Note however that the above factor in \( \Upsilon \) has degree \( m \cdot N \), hence, to preserve degree 0 component we can, for instance, add an element \( \prod_{i=1}^{N} z_{0,i}^{-m} \) that is compatible with the shuffle product.

For \( 0 \leq i \leq m - 1 \), we have:

specialization of \( \left( s_0 \ldots s_i \prod_{r=1}^{N} x_{i,r} - \mu \prod_{r=1}^{N} x_{i+1,r} \right) \) = const \( \cdot \prod_{r=1}^{N} z_{0,r} \)

For \( m \leq i \leq m + n - 1 \), we have:

specialization of \( \left( s_0 \ldots s_i \prod_{r=1}^{N} x_{i,r} - \mu \prod_{r=1}^{N} x_{i+1,r} \right) \) = const · \( s_0 \ldots s_i \prod_{r=1}^{N} z_{i-r,m,r} \cdot q^{-m} \prod_{r=1}^{N} z_{i-r+1,m+1,r} \)

Thus, we have

\[
\left( s_0 \ldots s_m \prod_{r=1}^{N} z_{0,r} - \mu \cdot q_3^{-mN} \prod_{r=1}^{N} z_{1,r} \right) \cdot \\
\left( s_0 \ldots s_m s_m+1 \prod_{r=1}^{N} z_{1,r} - \mu \cdot q_3^{-mN} \prod_{r=1}^{N} z_{2,r} \right) \cdot \\
\left( s_0 \ldots s_{n+m-1} \prod_{r=1}^{N} z_{n-1,r} - \mu \cdot q_3^{-mN} \prod_{r=1}^{N} z_{0,r} \right)
\]

Let \( \tilde{\Upsilon} = \Upsilon \cdot \prod_{i=1}^{N} z_{0,i}^{-m} \), we have \( \tilde{\Upsilon}_{m+n,n} : \bigoplus_{(k_0, \ldots, k_m)} Sh_{\mathfrak{gl}_{m+n}} \rightarrow Sh_{\mathfrak{gl}_n} \).

Therefore, \( \tilde{\Upsilon} (A(s_0, s_1, \ldots, s_{m+n-1})) = A(s_0, s_1, s_m, s_{m+1}, s_{m+n-1}) \subset Sh_{\mathfrak{gl}_n} \)

4.3 horizontal Heisenberg

**Theorem 4.4 Heisenberg Subalgebra Compatibility**

\( \tilde{\Upsilon} : (Heis_{\text{hor}}^+, \leq U(\mathfrak{gl}_{m+n})) = (Heis_{\text{hor}}^+, \leq U(\mathfrak{gl}_n)) \)

So, under \( \tilde{\Upsilon} \), we also have \( \tilde{\Upsilon} : (Heis_{m+n+}^+, \leq U(\mathfrak{gl}_{m+n})) = (Heis_{m+n+}^+, \leq U(\mathfrak{gl}_n)) \).
Proof.
This result is a direct consequence of Theorem 4.2 in the limit \( s_1, s_2, \ldots, s_{m+n-1} \to 0 \) of [N3].

4.5 Subalgebra of slope 0
Following the definition provided in [N3], we recall slope 0 subalgebra, \( A^0 \subset A^\infty \)
\( \{ F \in Sh_{ \vartheta_{m+n}} \mid \lim_{\xi \to \infty} F^t_{\xi} 0 \leq t \leq k \} \)

Let’s check if this condition behaves well under the map of \( \tilde{\Upsilon} \).

Proposition 4.6 Compatibility of Subalgebra of slope 0
\( \tilde{\Upsilon} \) is compatible with subalgebras of slope 0, that is \( \tilde{\Upsilon}(F) \in A^0_{\vartheta^l_n} \) for \( F \in A^0_{\vartheta_{m+n}} \).

Proof.
\( \tilde{\Upsilon} = (l_0 \ldots l_{n-1}) \leq (k_0 \ldots k_{n-1}) \) then specializing \( \{ z_{i, r}^0 \leq i \leq n-1, 1 \leq r \leq l_i \} \) to \( \xi \cdot z_{i, r}^0 \) with \( \xi \to \infty \) in \( \tilde{\Upsilon}(F) \) is equal to specialize \( \{ x_{i, r}^0 \mid m+1 \leq i \leq m, 1 \leq r \leq l_0 \} \cup \{ x_{i, r}^0 \mid m+1 \leq i \leq m+n-1, 1 \leq r \leq l_{i-m} \} \) to \( \xi \cdot x_{i, r}^0 \quad \xi \to \infty \).

We check the factor of \( t \) if it has limit as \( \xi \to \infty \).

\[
t = \frac{k_0 \prod_{r=1}^{n-1} (z_{0,r}^0 - q_3 z_{0,s}^0)^m \prod_{a=0}^{n-1} \prod_{1 \leq r \leq k_a} (z_{a,r}^0 - q_3 z_{a+1,s}^0)}{\prod_{a=0}^{n-1} \prod_{1 \leq r \leq k_a} (z_{a,r}^0 - z_{a+1,s}^0) \cdot \prod_{1 \leq r \neq s \leq k_0} (z_{0,r}^0 - q^2 z_{0,s}^0)^m \prod_{r=1}^{m} z_{0,r}^m}
\]

As \( \xi \to \infty \), the numerator of \( t \) approaches
\[
m(k_0^2 - (k_0 - l_0)^2) + \sum_{a=0}^{n-1} (k_a k_{a+1} - (k_a - l_a)(k_{a+1} - l_{a+1}))
\]

the denominator of \( t \) approaches
\[
m(k_0(k_0 - 1) - (k_0 - l_0)(k_0 - l_0 - 1)) + ml_0 + \sum_{a=0}^{n-1} (k_a k_{a+1} - (k_a - l_a)(k_{a+1} - l_{a+1}))
\]

Note that
\[
k_0^2 - (k_0 - l_0)^2 = 2k_0 l_0 - l_0^2
\]
\[
k_0(k_0 - 1) - (k_0 - l_0)(k_0 - l_0 - 1) + l_0 = 2k_0 l_0 - l_0^2
\]

Therefore, the factor of \( t \) has nonzero limit as we specialize \( z_{i, r}^0 \) to \( \xi \cdot z_{i, r}^0 \) when \( \xi \to \infty \). Under \( \tilde{\Upsilon} \), we have \( \tilde{\Upsilon}(A^0_{\vartheta_{m+n}}) \subseteq A^0_{\vartheta^l_n} \), which means \( \tilde{\Upsilon} \) is surjective.
Further directions

From our discussion of mappings between $Sh_{\mathfrak{gl}_{m+n}} \rightarrow Sh_{\mathfrak{gl}_n}$, two questions arise for further explorations: From our discussion of algebra homomorphisms connecting subalgebras of $Sh_{\mathfrak{gl}_{m+n}}$ with $Sh_{\mathfrak{gl}_n}$, two types:

1. Besides for subspaces/subalgebras $Sh(k_0 \ldots k_0, k_1 \ldots k_1)$, there are many more subalgebras coming from constants on degrees.

2. It’s a very interesting question to generalize the present result of $\Upsilon$ to the $Sh_\mathfrak{g}$ for simple Lie algebras of other types, in particular, to types $B_n, C_n, D_n$.

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