Specialization homomorphisms among shuffle algebras

Chuhan Wang

Abstract

We connect the shuffle algebras depending on a parameter n to each other via *specialization maps*. In particular, we get $n2^{n-1}$ of algebra homomorphisms from the *imaginary* subalgebra of the n-th shuffle algebra to the smallest one. We also verify their compatibility with the previously studied Bethe, Heisenberg, and slope 0 subalgebras. Our approach is "dual" to [FJMM] that used fused currents.

1 Introduction

1.1 Summary of results

Shuffle algebras were first introduced by Feigin and Odesskii to study elliptic quantum groups corresponding to elliptic curves, see [FO]. At the same time, the classical quantum groups arose as q-deformation of universal enveloping Lie algebras in the 1970s, and since then, they have had many applications in maths and physics, namely in fields such as string theory, quantum field theory, combinatorics, inverse scattering method, geometric representation theory, quantization, and deformation theory. In the standard numerous literature the quantum groups are explicitly defined by generators and a long list of defining relations. The three common realizations of the q-deformed loop Lie algebra include the Drinfel'd-Jimbo realization, the new Drinfeld/loop realization, and the so-called RTT realization (see [DF]). At the same time, as became clear over the last decade, the trigonometric version of the Feigin-Odesskii shuffle algebras provide a powerful combinatorial approach to quantum affine/loop groups, which is completely free of the cumbersome collections of generators and defining relations. In this realization, the algebras are realized in the space of multi-symmetric rational functions subject to some constraints on numerators and denominators, and with a rather simple but non-trivial shuffle product.

In the present paper, we connect various shuffle algebras among themselves in view of the above discussion, This can be viewed as a dual version of the construction of [FJMM] that embeds quantum toroidal (a.k.a. double affine/loop) $\mathfrak{gl}_1$ into the imaginary-degree part of the quantum toroidal $\mathfrak{gl}_n$ via "fused currents". Since the latter objects are subject to certain convergence issues and may be defined only on a suitable category of representations, our combinatorial construction simplifies and generalizes that of [FJMM] as well as allows to establish a good behavior with respect to previously studied various important subalgebras in question.

1.2 Outline of the paper

The structure of the paper is as follows. We recall the definitions of shuffle algebras and quantum affine algebras along with other key elements of our setup in section 2. In Section 3, we introduce the first two main results of the paper in Main and Second Theorems. Then, we detail the main construction of our homomorphism mapping from shuffle algebras one dimension to shuffle algebras of another dimension with examples. This is where we will perform the main calculations that will give us the new parameters for the algebra. In Section 4, we verify the compatibility of our algebra homomorphisms with some of the known subalgebras, such as Bethe algebras and horizontal Heisenberg algebras. In Section 5, we outline some questions for future pursuits in this topic.

2 Set Up

2.1 Lie Algebras and Simple Lie Algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space over an arbitrary field $\mathbb{F}$ with an operation:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called a Lie bracket, such that the following three properties hold:

1. This operation is bilinear: for $\forall x, y, z \in \mathfrak{g}, \quad \forall a, b \in \mathbb{F}$

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[x, ay + bz] = a[x, y] + b[x, z]$$

2. Skew-symmetry: $[x, y] = -[y, x] \quad \forall x, y \in \mathfrak{g}$

3. Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in \mathfrak{g}$

A Lie algebra $\mathfrak{g}$ is *finite-dimensional simple* if $\mathfrak{g}$ is not abelian and the only ideals in $\mathfrak{g}$ are 0 and $\mathfrak{g}$ itself. The study of simple Lie algebras was carried out in the last century. In particular, using the root decomposition, one can classify simple Lie algebras by the combinatorial data of Dynkin diagrams or equivalently Cartan matrices. The latter can be used to present simple $\mathfrak{g}$ by generators and relations.

2.2 Dynkin Diagram and Cartan Matrix

2.2.1 Dynkin Diagram

All simple Lie algebras are in bijection with one of such *Dynkin diagrams* and they are a way to encode roots of Lie algebras using nodes, therefore defining Lie algebras and providing insights into the representations of Lie groups. Each node in the diagram corresponds to a copy of $\mathfrak{sl}_2$. Two vertices are disconnected if the corresponding $\mathfrak{sl}_2$ commute and are joined by a single line whenever there is a nontrivial commutation relation. In this paper, we use I to represent the set of vertices of the Dynkin diagram of $\mathfrak{g}$. The Dynkin diagrams with only one edge are known as *simply laced Dynkin diagrams.*

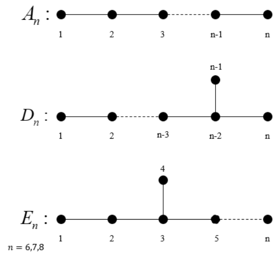

Figure 1: Dynkin diagrams for some simply-connected simple Lie algebras

Here, the other Dynkin Diagrams of types B, C, D, E, G are left out of the picture because they are not simply laced. Dynkin diagrams are especially simple when they are representing simply laced Lie algebras. They are simple to treat because they don't contain multiple edges, and the set of Serre relations are of either degree 2 or 3. In general setup, they can be much more complicated and the sub family of simply connected Lie algebras requires more a refined notion and all the combinatorial data about that.

A more specific way to construct Dynkin diagrams will be discussed below when we introduce Cartan Matrices.

2.2.2 Cartan matrix

In this paper, Cartan matrix will be denoted by $A = (a_{ij})_{i,j=1}^{n}$ with $n = |I|$ which is the number of vertices of the Dynkin diagram. A should be an symmetrizable $n \times n$ positive definite square matrix subject to the following relations:

$$\begin{cases} a_{ii} = 2 \\ a_{ij} \in \mathbb{Z}_{\leqslant 0} \quad \forall i \neq j \\ a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \end{cases}$$

We note that Cartan matrices of simple Lie algebras are diagonalizable, i.e., there exist relatively prime positive integers d_i such that $d_i a_{ij} = d_j a_{ji}$. The symmetrized Cartan matrix (d_{ij}) is defined to have entries given by these values $d_i a_{ij}$. A crucial property of the Cartan matrices is that it defines a positive definite quadratic form. That is, we have the inequality $\sum_{i,j=1}^{n} d_{ij} x_i x_j \geq 0$ for any $x_1, \ldots, x_n \in \mathbb{R}$ with an equality only for $x_1 =, \ldots, = x_n = 0$. For Cartan matrices of simply laced Dynkin Diagrams, all $d_i = 1$. For that reason, there's no real difference for d_{ij} and a_{ij} in simply laced cases. Replacing the condition of positive definite by positive semidefinite forms (where the equality holds not only for the zero vector) gives rise to affine/loop algebras that will be introduced below in a different realization.

2.3 Tensor Algebra

Let $V^{\otimes k}$ denote the k-fold tensor product of the vector space V, defined over a field $\mathbb{F}$. The *tensor algebra* $T(V)$ is defined to be the vector space

$$T(V) = \mathbb{F} \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

with sum between $V^{\otimes k}$ as direct sums and multiplication defined by external tensor product, namely $(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_m) = a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m$. Once a basis $\{e_1 \cdots e_n\}$ of V is chosen, $T(V)$ is naturally identified with the free associative algebra generated by$\{e_1 \cdots e_n\}$ with a multiplication being a concatenation of words.
If $\mathfrak{g}$ is a Lie algebra with a basis of $\{e_1 \cdots e_n\}$, the *universal enveloping algebra* $U(\mathfrak{g})$ is defined as the quotient of $T(\mathfrak{g})$ by two-sided ideal generated by $e_i \otimes e_j - e_j \otimes e_i - [e_i, e_j]$. This construction is actually independent of a choice of a basis, as follows from the universal property of $U(\mathfrak{g})$:

$$Hom_{\text{Lie algebra}}(\mathfrak{g}, A) = Hom_{\text{Associative algebra}}(U(\mathfrak{g}), A)$$

for any associative algebra A, endowed with a Lie bracket via $[a, b] = ab - ba$.

2.4 Loop Lie Algebras

For any Lie algebra $\mathfrak{g}$, the loop Lie algebra $\mathcal{L}\mathfrak{g}$ is defined as the vector space:

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \left\{ a(t) = \sum_n a_n t^n \middle| a_n \in \mathfrak{g}, n \in \mathbb{Z},\ a_n = 0 \text{ for } |n| \gg 0 \right\}$$

endowed with the Lie bracket:

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} \quad \forall x, y \in \mathfrak{g} \quad \forall n, m \in \mathbb{Z}$$

For example, if $\mathfrak{g} = \mathfrak{sl}_2$, the traceless 2×2 matrices which have the basis $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then the loop algebra $\mathcal{L}(\mathfrak{sl}_2)$ has a basis $\{et^n, ft^n, ht^n\}_{n \in \mathbb{Z}}$.

2.5 Drinfeld-Jimbo Quantum Group

We define the q-integers, factorials, and binomial coefficients via:

$$[k]_i = \frac{{q^{d_i}}^k - q^{d_i - k}}{q^{d_i} - {q^{d_i}}^{-1}}, \quad [k]\,!_i = [1]_i \ldots [k]_i, \quad \left(\begin{array}{c} n \\ k \end{array} \right)_i = \frac{[n]\,!_i}{[k]\,!_i\,[n-k]\,!_i}$$

For a finite-dimensional simple Lie algebra $\mathfrak{g}$, the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ is generated by $\left\{E_i^{\pm}, K_i^{\pm 1}\right\}_{i\in I}$, with I denoting the set of the vertices of the Dynkin diagram of $\mathfrak{g}$ as before and the following defining relations (for all $i, j \in I$).

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left(\begin{array}{c} 1-a_{ij} \\ k \end{array} \right)_i \left(E_i^{\pm}\right)^k E_j^{\pm} \left(E_i^{\pm}\right)^{1-a_{ij}-k} = 0, \quad i \neq j \tag{1}$$

$$K_j E_i^{\pm} = q^{\pm d_{ji}} E_j^{\pm} K_j \qquad K_i K_j = K_j K_i \tag{2}$$

$$\left[E_i^+, E_j^-\right] = \delta_i^j \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \tag{3}$$

Here, as agreed above, (a_{ij}) denotes the entry of Cartan matrix of $\mathfrak{g}$ and (d_{ij}) denotes that of the symmetrized Cartan matrix of $\mathfrak{g}$. The relations (1) are commonly called q-Serre relations.

2.6 Quantum Loop Algebra

The quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ is a quantization of the universal enveloping algebra of $\mathcal{L}\mathfrak{g}$. We can define it by generators $\left\{E_{i,r}^{\pm}, K_{i,l}^{\pm}, q^{\pm c}\right\}_{i\in I}^{r\in\mathbb{Z},\, l\in\mathbb{N}}$, satisfying the following relations. The relations are:

$$e_i^+(z)e_j^+(w)\left(w - zq^{-d_{ij}}\right) = e_j^+(w)e_i^+(z)\left(wq^{-d_{ij}} - z\right) \tag{4}$$

$$e_i^-(z)e_j^-(w)\left(w - zq^{d_{ij}}\right) = e_j^-(w)e_i^-(z)\left(wq^{d_{ij}} - z\right) \tag{5}$$

$$\begin{gathered} \text{if } i \neq j \quad \textstyle\sum_{\sigma\in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \left(\begin{array}{c} 1-a_{ij} \\ k \end{array} \right)_i \\ e_i^+\left(z_{\sigma(1)}\right)\dots e_i^+\left(z_{\sigma(k)}\right) e_j^+(w) e_i^+\left(z_{\sigma(k+1)}\right)\dots e_i^+\left(z_{\sigma(1-a_{ij})}\right) = 0 \end{gathered} \tag{6}$$

$$\begin{gathered} \text{if } i \neq j \quad \textstyle\sum_{\sigma\in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \left(\begin{array}{c} 1-a_{ij} \\ k \end{array} \right)_i \\ e_i^-\left(z_{\sigma(1)}\right)\dots e_i^-\left(z_{\sigma(k)}\right) e_j^-(w) e_i^-\left(z_{\sigma(k+1)}\right)\dots e_i^-\left(z_{\sigma(1-a_{ij})}\right) = 0 \end{gathered} \tag{7}$$

$$K_j^{\pm}(w)e_i^+(z)\left(z - wq^{-d_{ij}}\right) = e_i^+(z)K_j^{\pm}(w)\left(zq^{-d_{ij}} - w\right) \tag{8}$$

$$K_j^{\pm}(w)e_i^-(z)\left(z - wq^{d_{ij}}\right) = e_i^-(z)K_j^{\pm}(w)\left(zq^{d_{ij}} - w\right) \tag{9}$$

$$K_i^{\pm}(z)K_j^{\pm'}(w) = K_j^{\pm'}(w)K_i^{\pm}(z), \quad K_{i,0}^+ K_{i,0}^- = 1 \tag{10}$$

$$\left[e_i^+(z), e_j^-(w)\right] = \frac{\delta_i^j \delta\left(\frac{z}{w}\right)}{q_i - q_i^{-1}} \cdot \left(K_i^+(z) - K_i^-(w)\right) \tag{11}$$

where $q_i = q^{d_i}$. We first define the currents as follows:

$$e_i^+(z) = \sum_{r\in\mathbb{Z}} E_{i,r}^+ z^{-r}, \quad e_i^-(z) = \sum_{r\in\mathbb{Z}} E_{i,r}^- z^{-r}, \quad K_i^{\pm}(z) = \sum_{l\in\mathbb{N}} K_{i,l}^{\pm} z^{\mp l}$$

We note that the assignment $E_i^+ \mapsto E_{i,0}^+$, $E_i^- \mapsto E_{i,0}^-$ and $K_i^\pm \mapsto K_{i,0}^\pm$ gives rise to a injective algebra homomorphism : $U_q(\mathfrak{g}) \to U_q(\mathcal{L}\mathfrak{g})$. Evoking the triangular decomposition:

$$U_q(\mathcal{L}\mathfrak{g}) \cong U_q(\mathcal{L}\mathfrak{g})^+ \otimes U_q(\mathcal{L}\mathfrak{g})^0 \otimes U_q(\mathcal{L}\mathfrak{g})^- \tag{12}$$

where the middle factor is commutative, while the third factor is naturally isomorphic to the first one via $E_{i,r}^+ \mapsto E_{i,-r}^-$, we shall just focus on the positive subalgebra $U_q(\mathcal{L}\mathfrak{g})^+$ generated by $E_{i,r}^+$ with $i \in I, r \in \mathbb{Z}$.

2.7 Quantum Toroidal Algebras of $\mathfrak{gl}_n$

The crucial feature of the loop algebras $\mathcal{L}\mathfrak{g}$ for simple $\mathfrak{g}$ is that they can also be defined by generators and relations, similarly to $\mathfrak{g}$, but starting from the extended Dynkin diagram or extended Cartan matrix corresponding to positive semidefinite quadratic forms, see 2.2. As such taking the loops in them, one obtains the toroidal algebras $U_q(\widehat{\mathfrak{g}})$. However, only in type A_n, it is also possible to introduce yet another deformation parameter, which corresponds to having cycles in the corresponding extended Dynkin diagram. This results in the quantum toroidal algebras $U_{q,d}(\widehat{\widehat{\mathfrak{gl}}}_n)$ that underline the present work. In the rest of the paper we shall however use their combinatorial realizations, evoked in the Introduction, which we recall in the next section.

2.8 Shuffle algebras

Shuffle algebras were first introduced by Feigin and Odesskii to study elliptic quantum groups corresponding to elliptic curves, see [FO]. Let Σ_k denote the symmetric group of k elements, and $\Sigma_{(k_1,\dots,k_n)} = \Sigma_{k_1} \times \cdots \times \Sigma_{k_n}$ for $k_1, \dots, k_n \in \mathbb{N}$. Consider an $\mathbb{N}^I$-graded vector space $\mathbb{S}^{(n)} = \bigoplus \mathbb{S}_{\underline{k}}^{(n)}$, where $\underline{k} = (k_1, \dots, k_n) \in \mathbb{N}^I$ and $\mathbb{S}_{\underline{k}}^{(n)}$ consists of $\Sigma_{\underline{k}}$ symmetric rational functions in the variables $\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}$. Here, $n = |I|$ as before. In this paper, we use $X_{i,1}^{i,k_i}$ to represent $\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}$.

For any collection of rational functions $\zeta_{ij}(z)$, we endow the shuffle algebra $\mathbb{S}^{(n)}$ with the following bilinear shuffle product $\star$: given $F \in \mathbb{S}_{\underline{k}}^{(n)}$ and $G \in \mathbb{S}_{\underline{l}}^{(n)}$ define $F \star G \in \mathbb{S}_{\underline{k}+\underline{l}}^{(n)}$

$$(F \star G)\left(X_{1,1}^{1,k_1+l_1}, \dots, X_{n-1,1}^{n-1,k_{n-1}+l_{n-1}}\right) = \frac{1}{\underline{k}! \cdot \underline{l}!} \cdot$$
$$\mathrm{Sym}_{\Sigma_{\underline{k}+\underline{l}}}\left(F\left(X_{i,1}^{i,k_i}\right) \cdot G\left(X_{j,k_j+1}^{j,k_j+l_j}\right) \cdot \prod_{i\in I}^{j\in I} \prod_{r\le k_i}^{s>k_j} \zeta_{i,j}\left(\frac{x_{i,r}}{x_{j,s}}\right)\right)$$

where $\underline{k}! = \prod_{i\in I} k_i!$, and the symmetrization of $F \in \mathbb{C}\left(X_{i,1}^{i,k_i}\right)$ is defined via

$$\mathrm{Sym}_{\Sigma_{\underline{k}}}(F)\left(X_{i,1}^{i,k_i}\right) = \sum_{(\sigma_1 \dots \sigma_{n-1}) \in \Sigma_{\underline{k}}} F\left(X_{i,\sigma_i(1)}^{i,\sigma_i(k_i)}\right)$$

, the averaging of F over the product of symmetric groups acting termwise by permutation of the variables of each color.

In the following four sections 2.8.1–2.8.4, we introduce the shuffle factors as well as pole and wheel conditions for all $_n$ with $n \geq 1$. While the treatments of n=1, n=2, n=3, and n>3 cases are different, they naturally fit into the same framework and have a common underlying picture and logic.

2.8.1 Shuffle Algebra for $\mathfrak{gl}_1$

If $\mathfrak{g} = \mathfrak{gl}_1$, since there is only one color, we let $k_1 = k$, $\{x_{1,r}\}_{r=1}^{k} = \{x_r\}_{r=1}^{k}$. If $f\left(X_1^k\right) = \frac{F\left(X_1^k\right)}{\prod\limits_{1 \leq i \neq j \leq k} (x_i - x_j)}$, and F is a symmetric Laurent polynomial, then f is said to satisfy the pole condition.
If $f\left(X_1^k\right) = 0$ whenever there exist 1 a, b, c k such that $x_a = qd^{-1}x_b, x_b = qdx_c$ and $x_c = q^{-2}x_a$, then f is said to satisfy the wheel condition. These ratios are conveniently encoded by the weighted edges as depicted in Figure 2
Let $Sh_{\widehat{\mathfrak{gl}}_1}(k)$ denote the subspace of all $f \in \mathbb{S}$ satisfying both pole and wheel conditions.

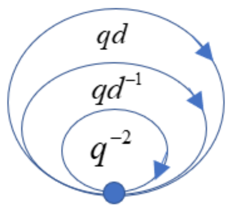

Figure 2: Wheel condition-$Sh_{\widehat{\mathfrak{gl}}_1}$

The shuffle factor of $Sh_{\widehat{\mathfrak{gl}}_1}$ is given by

$$\omega^{(1)}\left(\frac{x_i}{x_j}\right) = \frac{(x_i - q^{-2}x_j)(x_i - qd^{-1}x_j)(x_i - qdx_j)}{(x_i - x_j)^3}$$
$$= \frac{(x_i - q_2^{-1}x_j)(x_i - q_1^{-1}x_j)(x_i - q_3^{-1}x_j)}{(x_i - x_j)^3}$$

2.8.2 Shuffle Algebra for $\mathfrak{gl}_2$

If $\mathfrak{g} = \mathfrak{gl}_2$, we let $\{x_{1,r}\}_{r=1}^{k_1} = X_{1,1}^{1,k_1}$ and $\{x_{2,s}\}_{s=1}^{k_2} = X_{2,1}^{2,k_2}$, specify $X_{1,1}^{1,k_1}$ color 0, and $X_{2,1}^{2,k_1}$ color 1.
If $f\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}\right) = \frac{F\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}\right)}{\prod\limits_{1 \leq r \leq k_1}^{1 \leq s \leq k_2} (x_{1,r} - x_{2,s})^2}$, and F is a symmetric Laurent polynomial, then f is said to satisfy the pole condition.
If $f\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}\right) = 0$ whenever there exist $1 \leq a, c \leq k_1$ and $1 \leq b \leq k_2$ meet one of following condition 1 or condition 2, then f is said to satisfy the wheel condition and we say f wheel in color $(0, 0, 1)$.

$$\begin{array}{llll} \text{condition 1:} & x_{1,a} = qdx_{2,b} & \text{and} & x_{2,b} = qd^{-1}x_{1,c} \\ \text{condition 2:} & x_{1,a} = qd^{-1}x_{2,b} & \text{and} & x_{2,b} = qdx_{1,c} \end{array}$$

If $f\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}\right) = 0$ whenever there exist $1 \le b \le k_1$ and $1 \le a, c \le k_2$ meet one of following condition 1 or condition 2, then f is said to satisfy the wheel condition, and we say f wheel in color $(1, 1, 0)$.

$$\begin{array}{llll} \text{condition 1:} & x_{2,a} = qdx_{1,b} & \text{and} & x_{1,b} = qd^{-1}x_{2,c} \\ \text{condition 2:} & x_{2,a} = qd^{-1}x_{1,b} & \text{and} & x_{1,b} = qdx_{2,c} \end{array}$$

These ratios are conveniently encoded by the weighted edges as depicted in Figure 3.

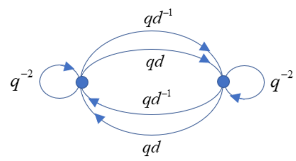

Figure 3: Wheel conditon-$Sh_{\widehat{\mathfrak{gl}}_2}$

Let $Sh_{\widehat{\mathfrak{gl}}_2}(k_1, k_2)$ denote the subspace of all $f \in \mathbb{S}$ satisfying both pole and wheel conditions. The shuffle factor of $Sh_{\widehat{\mathfrak{gl}}_2}$ is given by

$$\omega_{i,j}^{(2)}\left(\frac{x_{i,r}}{x_{j,s}}\right) = \begin{cases} \dfrac{x_{i,r} - q^{-2}x_{j,s}}{x_{i,r} - x_{j,s}}, & \text{if } j \equiv i \bmod 2; \\ \dfrac{(x_{i,r} - qd^{-1}x_{j,s})(x_{i,r} - qdx_{j,s})}{(x_{i,r} - x_{j,s})^2}, & \text{if } j \equiv i \pm 1 \bmod 2; \end{cases}$$

That is:

$$\omega_{i,j}^{(2)}\left(\frac{x_{i,r}}{x_{j,s}}\right) = \begin{cases} \dfrac{x_{i,r} - q_2^{-1}x_{j,s}}{x_{i,r} - x_{j,s}}, & \text{if } j \equiv i \bmod 2; \\ \dfrac{(x_{i,r} - q_1^{-1}x_{j,s})(x_{i,r} - q_3^{-1}x_{j,s})}{(x_{i,r} - x_{j,s})^2}, & \text{if } j \equiv i \pm 1 \bmod 2; \end{cases}$$

2.8.3 Shuffle Algebra for $\mathfrak{gl}_3$

If $\mathfrak{g} = \mathfrak{gl}_3$, we let $\{x_{1,r}\}_{r=1}^{k_1} = X_{1,1}^{1,k_1}$, $\{x_{2,s}\}_{s=1}^{k_2} = X_{2,1}^{2,k_2}$ and $\{x_{3,t}\}_{t=1}^{k_3} = X_{3,1}^{3,k_3}$, specify $X_{1,1}^{1,k_1}$ color 0, $X_{2,1}^{2,k_2}$ color 1 and $X_{3,1}^{3,k_3}$ color 2.

If $f\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}|X_{3,1}^{3,k_3}\right) = \frac{F\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}|X_{3,1}^{3,k_3}\right)}{\prod\limits_{1\le i\le k_1}^{1\le j\le k_2}(x_{1,i}-x_{2,j})\cdot\prod\limits_{1\le i\le k_2}^{1\le j\le k_3}(x_{2,i}-x_{3,j})\cdot\prod\limits_{1\le i\le k_3}^{1\le j\le k_1}(x_{3,i}-x_{1,j})}$, and F is a symmetric Laurent polynomial, then f is said to satisfy the pole condition.

If $f\left(X_{1,1}^{1,k_1}|X_{2,1}^{2,k_2}|X_{3,1}^{3,k_3}\right)=0$ whenever there exist $1\le a,c\le k_i$ and $1\le b\le k_j$ meet one of following condition 1 or condition 2, then f is said to satisfy the wheel condition and we say f wheel in color $(i-1,i-1,j-1)$, where $1\le i,j\le 3$.

$$\begin{aligned}&\text{condition 1:}\quad j\equiv i+1 \bmod 3,\quad x_{i,a}=qdx_{j,b}\quad &&\text{and}\quad x_{j,b}=qd^{-1}x_{i,c}\\ &\text{condition 2:}\quad j\equiv i-1 \bmod 3,\quad x_{i,a}=qd^{-1}x_{j,b}\quad &&\text{and}\quad x_{j,b}=qdx_{i,c}\end{aligned}$$

Let $Sh_{\widehat{\mathfrak{gl}}_3}(k_1,k_2,k_3)$ denote the subspace of all $f\in\mathbb{S}$ satisfying both pole and wheel conditions. The shuffle factor of $Sh_{\widehat{\mathfrak{gl}}_3}$ is given by

$$\omega_{i,j}^{(3)}\left(\frac{x_{i,r}}{x_{j,s}}\right)=\begin{cases}\dfrac{x_{i,r}-q^{-2}x_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i \bmod 3;\\ \dfrac{d^{-1}x_{i,r}-qx_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i+1 \bmod 3;\\ \dfrac{x_{i,r}-qd^{-1}x_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i-1 \bmod 3;\end{cases}$$

2.8.4 Shuffle Algebra for $\mathfrak{gl}_n$

If $\mathfrak{g}=\mathfrak{gl}_n$, we let $\{x_{1,r_1}\}_{r_1=1}^{k_1}=X_{1,1}^{1,k_1},\dots,\{x_{n,r_n}\}_{r_n=1}^{k_n}=X_{n,1}^{n,k_n}$, and specify $X_{1,1}^{1,k_1}$ color $0,\dots,X_{n,1}^{n,k_n}$ color $n-1$.

$$\text{If } f\left(X_{1,1}^{1,k_1},\dots,X_{n,1}^{n,k_n}\right)=\frac{F\left(X_{1,1}^{1,k_1},\dots,X_{n,1}^{n,k_n}\right)}{\prod\limits_{\substack{1\le j\le k_2\\1\le i\le k_1}}(x_{1,i}-x_{2,j})\cdot\prod\limits_{\substack{1\le j\le k_3\\1\le i\le k_2}}(x_{2,i}-x_{3,j})\dots\prod\limits_{\substack{1\le j\le k_1\\1\le i\le k_n}}(x_{n,i}-x_{1,j})},$$

and F is a symmetric Laurent polynomial, then f is said to satisfy the pole condition.
If $f\left(X_{1,1}^{1,k_1},\dots,X_{n,1}^{n,k_n}\right)=0$ whenever there exist $1\le a,c\le k_i$ and $1\le b\le k_j$ meet one of following condition 1 or condition 2, then we say the shuffle algebra wheel in color $(i-1,i-1,j-1)$, where $1\le i,j\le n$.

$$\begin{aligned}&\text{condition 1:}\quad j\equiv i+1 \bmod n,\quad x_{i,a}=qdx_{j,b}\quad &&\text{and}\quad x_{j,b}=qd^{-1}x_{i,c}\\ &\text{condition 2:}\quad j\equiv i-1 \bmod n,\quad x_{i,a}=qd^{-1}x_{j,b}\quad &&\text{and}\quad x_{j,b}=qdx_{i,c}\end{aligned}$$

These ratios are conveniently encoded by the weighted edges as depicted in Figure 4 Let $Sh_{\widehat{\mathfrak{gl}}_n}(k_1,\dots,k_n)$ denote the subspace of all $f\in\mathbb{S}^k$ satisfying both pole and wheel conditions. For $1\le r\le k_i$ and $1\le s\le k_j$, the shuffle factor of $Sh_{\widehat{\mathfrak{gl}}_n}$ is given by

$$\omega_{i,j}^{(n)}\left(\frac{x_{i,r}}{x_{j,s}}\right)=\begin{cases}\dfrac{x_{i,r}-q^{-2}x_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i \bmod n;\\ \dfrac{d^{-1}x_{i,r}-qx_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i+1 \bmod n;\\ \dfrac{x_{i,r}-qd^{-1}x_{j,s}}{x_{i,r}-x_{j,s}}, & \text{if } j\equiv i-1 \bmod n;\\ 1, & \text{otherwise.}\end{cases}$$

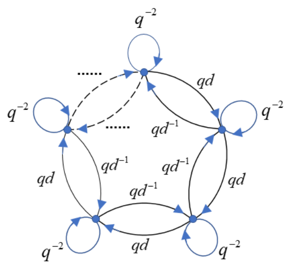

Figure 4: Wheel condition-$Sh_{\widehat{\widehat{\mathfrak{gl}}}_n}$

The relation between the shuffle and quantum toroidal algebras is established by the following results.

Proposition 2.9 *The assignment* $E^+_{i,r} \mapsto x^r_{i,1}$ *give rise to an algebra homomorphism* $\rho\colon U_{q,d}(U^{>}_{q,d}(\widehat{\widehat{\mathfrak{gl}}}_n))^{>} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_n}$

The proof of this result is straightforward as it requires only verifying that the above assignment is compatible with the defining quadratic and cubic relations. The following result is much harder and was proved in [N1, N2]

Theorem 2.10 *The homomorphism* ρ *in the Proposition 2.9 is actually an algebra isomorphism.*

3 Main Results

The paper of [FJMM] provided an interesting but quite obscure construction of the algebra homomorphisms from the quantum toroidal of $\mathfrak{gl}_1$ into the imaginary degree part of the quantum toroidal $\mathfrak{gl}_n$. The latter involved fusion of currents, which involves infinite sums, hence being defined only up to convergence assumptions.

We shall start our treatment from its simplest counterpart, namely the simplest non-trivial case, the map from $\bigoplus_k Sh_{\widehat{\widehat{\mathfrak{gl}}}_2} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$. This simplest case contains many of the key elements of our general construction and provides insight into the homomorphism, and how different inputs lead to predictable numbers of specializations. One of the goals is to construct an algebra homomorphism from the imaginary subalgebra of a shuffle algebra to the whole of another, mathematically speaking it is:
$Sh^{im}_{\widehat{\widehat{\mathfrak{gl}}}_n} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_m}$. For our main theorem, we generalize the discussion in [FHHSY] and construct algebra homomorphisms from certain subalgebras of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}$ to $Sh_{\mathfrak{gl}_n}$. There's a family of algebra homomorphisms given by certain specialization procedure that would be described later:

Theorem 3.1 (Main Theorem)

We have algebra homomorphism:

$$\Upsilon_{m+n,n} : \bigoplus_{k_0,k_1,\dots,k_{n-1}\in\mathbb{N}} Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}(\underbrace{k_0,\dots,k_0}_{m+1},\underbrace{k_1,\dots,k_{n-1}}_{n-1})|_{q_1,q_2,q_3} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_n}|_{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3}$$

Theorem 3.2 (Second Theorem)

We have algebra homomorphism:

$$\Upsilon_n : \bigoplus_{k\in\mathbb{N}} Sh_{\widehat{\widehat{\mathfrak{gl}}}_n}(\underbrace{k,\dots,k}_{n})|_{q_1,q_2,q_3} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}|_{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3}$$

Remark 3.3 *We note that the above two algebras both in Main Theorem and Second Theorem will have different parameters as the homomorphism doesn't preserve the parameters and there are different parameters q_1, q_2, q_3 and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ for different mappings depending on specific homomorphism constructions. The mapping is component-wise*

We will verify in Section 4 that these algebra homomorphisms with some compatibility of subalgebras studied in [N1] [FT]. In Section 4, we shall also verify that these homomorphisms are compatible with important subalgebras studied in [FHHSY, FT, N1].

3.4 From $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}$ to $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$

By considering the typical of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}$ to $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$, we can see that our maps of specialization will be of the following nature of specializing a segment of the variables and then multiplying by some rational function. From now on, we shall always define some specialization of a function f as $spec \cdot f$. In this case, we send f to its specialization times G, $f \mapsto spec(f) \cdot G$. Here, the specialization action is taking some of the variables to be multiples of each other and g as some rational function factor. See the later section for more details.
In $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,k)$, we have variables of $X_{1,1}^{1,k}$ and $X_{2,1}^{2,k}$ in two different colors. To simplify notations, we rename $X_{1,1}^{1,k}$ and $X_{2,1}^{2,k}$ by X_1^k and Y_1^k respectively. Further, in this paper, we use A_i^j to represent $(a_i, a_{i+1} \dots a_j)$, where a_i is a variable, and use cA_i^j to represent $(ca_i, ca_{i+1} \dots ca_j)$, where c is a constant.
In order to construct an algebra homomorphism $\Upsilon_{2,1} : Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,k) \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}(k)$ the general idea of our algorithm is: for $\forall\, f(X_1^k|Y_1^k) \in Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,k)$ we specialize $X_1^k = \lambda Y_1^k$, therefore, $\Upsilon_{2,1}(f(X_1^k|Y_1^k)) = f(\lambda Y_1^k|Y_1^k)$.
We know $f(X_1^k|Y_1^k)$ satisfies the pole condition and wheel condition of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}$, while $f(\lambda Y_1^k|Y_1^k)$ satisfies those conditions of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$, we can amend this by using a rational factor of $G(Y_1^k)$.

In addition, we also anticipate some constants that inherently come with the specializations, which we would denote c_k for a function $f({X_1}^k)$. Our goal is to determine the rational factor $G(Y_1^k)$, c_k, and explicit relations between q_1, q_2, q_3 and $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3$ that would make $\Upsilon_{2,1}$ a compatible homomorphism mapping.

Remark 3.5 *Our main argument for why $\lambda \in \{qd^{\pm 1}, qd^{\pm 1}\}$ comes from analyzing the number of terms in both sides of the homomorphism. We have homomorphism, $\Upsilon(f(x) \star g(y))$, for f, g functions of k, l variables respectively is $C_{k+l}^k \cdot C_{k+l}^k$, and such choices of λ to kill the unbalanced terms. On the LHS we are picking a set of k elements of a specific color to go into $f(x)$ from all elements of that color from $f(x)$ and $g(y)$, yielding C_{k+l}^k terms. On the RHS, $\Upsilon f(x) \star \Upsilon g(x)$ only has two terms. Therefore, make the number of terms on both sides equal by letting the ω terms on the left vanish, restricting the possible choices of λ to qd, qd^{-1}.*

For example. For $f(x_1|y_1), g(x_2|y_2) \in Sh_{\widehat{\mathfrak{gl}}_2}(1,1)$, let $x_i = \lambda y_i$ with $i = 1, 2$. To simplify the notations, we omit the shuffle factors and G factors in computing $f \star g$ by using the symbol $\sim$ instead of equality. To ensure 2,1(f g) = 2,1(f) 2,1(g), we must kill the terms containing f(y2/y1) · g(y1/y2) and f(y1/y2) · g(y2/y1). Tracing back the zeroes of the corresponding $\omega^{(2)}$-factors we find that = qd or = qd1. We want to construct a homomorphism $\Upsilon_{2,1}$ such that $\Upsilon_{2,1}(f \star g) \sim \Upsilon_{2,1}(f) \star \Upsilon_{2,1}(g)$.

$$
\begin{aligned}
\Upsilon_{2,1}(f \star g) \sim & f(\lambda y_1|y_1) \cdot g(\lambda y_2|y_2) + f(\lambda y_2|y_2) \cdot g(\lambda y_1|y_1) + \\
& f(\lambda y_2|y_1) \cdot g(\lambda y_1|y_2) + f(\lambda y_1|y_2) \cdot g(\lambda y_2|y_1) \\
\Upsilon_{2,1}(f) \star \Upsilon_{2,1}(g) \sim & f(y_1) \star g(y_2) \sim f(y_1) \cdot g(y_2) + f(y_2) \cdot g(y_1)
\end{aligned}
$$

To ensure $\Upsilon_{2,1}(f \star g) \sim \Upsilon_{2,1}(f) \star \Upsilon_{2,1}(g)$, we must kill the item of $f(\lambda y_2|y_1) \cdot g(\lambda y_1|y_2)$ and $f(\lambda y_1|y_2) \cdot g(\lambda y_2|y_1)$, which can be done by specializing $\lambda \in \{qd^{\pm 1}, qd^{\pm 1}\}$.

In addition, we can prove the validity of our λ choices by looking at the wheel conditions. We specify X_1^k color 0 and Y_1^k color 1. To map two colors to one single color, we have to make sure that the image of every shuffle product between two variables of different colors should be contained in $Sh_{\widehat{\mathfrak{gl}}_1}(k)$. This means $\omega^{(2)}_{i \neq j}(\frac{x}{y}) = 0$, so $(x - qd^{-1}y)(x - qdy) = 0$, implying that $x = qd^{-1}y$ or $x = qdy$ and $\lambda = qd$ or $\lambda = qd^{-1}$.

In summary, these λ ensure that we kill unwanted wheel conditions that arose from different colors but also preserve the homomorphism of the mapping.

Below, we will discuss how choices of λ yields different rational factor G and different relation between the parameters q_1, q_2, q_3 and $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3$:

3.5.1 Case 1: $\lambda = qd$

In this case, $X_1^k = qdY_1^k$. First, since $\Upsilon_{2,1}$ is an algebra homomorphism from $Sh_{\widehat{\mathfrak{gl}}_2}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$, we must have $\prod\limits_{1\le i\ne j\le k}(y_i - y_j)$ in the denominator, and have $\prod\limits_{1\le i,j\le k}(qdy_i - y_j)^2$ in the numerator to eliminate the denominator of $f(qdY_1^k|Y_1^k)$.
Second, the specialization of f has certain zero divisors due to the wheel conditions of $Sh_{\widehat{\mathfrak{gl}}_1}$. Explicitly, we have zeroes at any $y_i - q^2y_j$ for $i \ne j$. Thus, we try

$$G(Y_1^k) = \frac{\prod\limits_{1\le i,j\le k}(qdy_i - y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i - y_j)^2(y_i - q^2y_j)(y_i - q^{-2}y_j)} \cdot C_k$$

C_k here is a constant that is yet to be determined, which depends on q and d but not Y_1^k. In particular, when $k = 1$,

$$\Upsilon_{2,1}(f(x\,|y)) = \frac{p(qdy\,|y)}{(qdy - y)^2} \cdot (qdy - y)^2 \cdot C_1 = p(qdy\,|y) \cdot C_1$$

Given $f(X_1^k|Y_1^k) \in Sh_{\widehat{\mathfrak{gl}}_2}(k,k)$ and $g(X_{k+1}^{k+l}|Y_{k+1}^{kl}) \in Sh_{\widehat{\mathfrak{gl}}_2}(l,l)$, we have

$$\Upsilon_{2,1}(f \star g) = Sym\left[\begin{array}{c} f\left(qdY_1^k|Y_1^k\right) \cdot g\left(qdY_{k+1}^{k+l}|Y_{k+1}^{k+l}\right) \cdot \omega_{i,i}^{(2)}\left(\dfrac{qdY_1^k}{qdY_{k+1}^{k+l}}\right) \\ \cdot\omega_{i,i+1}^{(2)}\left(\dfrac{qdY_1^k}{Y_{k+1}^{k+l}}\right) \cdot \omega_{i,i-1}^{(2)}\left(\dfrac{Y_1^k}{qdY_{k+1}^{k+l}}\right) \cdot \omega_{i,i}^{(2)}\left(\dfrac{Y_1^k}{Y_{k+1}^{k+l}}\right) \end{array}\right].$$

$$\frac{\prod\limits_{1\le i,j\le k+l}(qdy_i - y_j)^2}{\prod\limits_{1\le i<j\le k+l}(y_i - y_j)^2(y_i - q^2y_j)(y_i - q^{-2}y_j)} \cdot C_{k+l}$$

To simplify the notations, let $\omega_{i,j}^{(n)}(\frac{Y_a^b}{Y_c^d}) = \prod\limits_{a\le i\le b}^{c\le j\le d}\omega_{i,j}^{(n)}\left(\frac{y_i}{y_j}\right)$, that is

$$\omega_{i,i}^{(2)}(\frac{qdY_1^k}{qdY_{k+1}^{k+l}}) = \prod_{1\le i\le k}^{k+1\le j\le k+l}\left(\frac{qdy_i - qdq^{-2}y_j}{qdy_i - qdy_j}\right) = \prod_{1\le i\le k}^{k+1\le j\le k+l}\left(\frac{y_i - q^{-2}y_j}{y_i - y_j}\right)$$

$$\omega_{i,i+1}^{(2)}(\frac{qdY_1^k}{Y_{k+1}^{k+l}}) = \prod_{1\le i\le k}^{k+1\le j\le k+l}\left(\frac{qdy_i - qd^{-1}y_j}{qdy_i - y_j}\right)\left(\frac{qdy_i - qdy_j}{qdy_i - y_j}\right)$$

$$\omega_{i,i-1}^{(2)}(\frac{Y_1^k}{qdY_{k+1}^{k+l}}) = \prod_{1\le i\le k}^{k+1\le j\le k+l}\left(\frac{y_i - q^2d^2y_j}{y_i - qdy_j}\right)\left(\frac{y_i - q^2y_j}{y_i - qdy_j}\right)$$

$$\omega_{i,i}^{(2)}(\frac{Y_1^k}{Y_{k+1}^{k+l}}) = \prod_{1\le i\le k}^{k+1\le j\le k+l}\left(\frac{y_i - q^{-2}y_j}{y_i - y_j}\right)$$

We can move the latter fraction of $\Upsilon_{2,1}(f \star g)$ inside the Sym bracket. After cancellation of common terms between the numerator and the denominator, we get

$$\Upsilon_{2,1}(f\star g) = Sym\left[\begin{array}{l}\Upsilon(f(Y_1^k))\cdot\Upsilon(g(Y_{k+1}^{k+l}))\cdot\\ \prod\limits_{1\le i\le k}^{k<j\le k+l}\dfrac{(y_i-q^{-2}y_j)(y_i-q^2d^2y_j)(y_i-d^{-2}y_j)}{(y_i-y_j)^3}\end{array}\right]\cdot\frac{C_{k+l}}{C_kC_l}\cdot(qd)^{-2kl}$$

Picking $c_k=(qd)^{k^2}$ or wherever, we get Formula for G follows:

$$G=\frac{\prod\limits_{1\le i,j\le k}(qd^{-1}y_i-y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}\cdot(qd)^{-k^2}$$

with the parameters $\tilde{q}_1,\tilde{q}_2,\tilde{q}_3$ related to q,d via

$$\widetilde{q}_1=d^2,\widetilde{q}_2=q^2,\widetilde{q}_3=q^{-2}d^{-2}$$

we construct an algebra homomorphism,

$$\Upsilon_{2,1}:\bigoplus_k Sh_{\widehat{\mathfrak{gl}}_2}(k,k)|_{q_1,q_2,q_3}\to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1=q_1q_3^{-1},\widetilde{q}_2=q_2,\widetilde{q}_3=q_3^2}\qquad\text{given by}$$

$$\Upsilon_{2,1}:f\left(X_1^k|Y_1^k\right)\to f\left(qdY_1^k|Y_1^k\right)\cdot\frac{(qd)^{-k^2}\cdot\prod\limits_{1\le i,j\le k}(qdy_i-y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}$$

Remark 3.6 *As we saw above, the verification of the fact that $\Upsilon_{2,1}$ was an algebra homomorphism eventually boiled down to a solution for the system $\frac{c_{k+l}}{c_kc_l}=\beta_{k,l}$ with the right-hand sides explicitly give. The existence of such c_k is guaranteed by the validity of the equalities $\beta_{k+l,m}\beta_{k,l}=\beta_{k,l+m}\beta_{l,m}$ for any $k,l,m\ge 1$ In the above example, we can easily see that $c_k=(qd)^{-k^2}$ is a solution.*

In the end, the existence of such constant is given by **??**, but every constant is given as a fraction in the sentence below:

$$\frac{c_{k+l+m}}{c_{k+l}c_m}\cdot\frac{c_{k+l}}{c_kc_l}=\frac{c_{k+l+m}}{c_kc_{l+m}}\cdot\frac{c_{l+m}}{c_lc_m}\tag{13}$$

3.6.1 Case 2: $\lambda=qd^{-1}$

We specialize $X_1^k=qd^{-1}Y_1^k$.
Similar to the analysis of 3.5.1, we have

$$G(Y_1^k)=\frac{\prod\limits_{1\le i,j\le k}(qd^{-1}y_i-y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}\cdot C_k$$

Again, we have to check whether the products are compatible and determine $G(Y_1^k)$, $\widetilde{q}_1$, $\widetilde{q}_2$ and $\widetilde{q}_3$.

$$\Upsilon_{2,1}(f\star g)=Sym\left[\begin{array}{c} f(qd^{-1}Y_1^k\left|Y_1^k\right.)\cdot g(qd^{-1}Y_{k+1}^{k+l}|Y_{k+1}^{k+l})\cdot\omega_{i,i}^{(2)}\left(\frac{Y_1^k}{Y_{k+1}^{k+l}}\right)\cdot \\ \omega_{i,i+1}^{(2)}\left(\frac{qd^{-1}Y_1^k}{Y_{k+1}^{k+l}}\right)\cdot\omega_{i,i-1}^{(2)}\left(\frac{Y_1^k}{qd^{-1}Y_{k+1}^{k+l}}\right)\cdot\omega_{i,i}^{(2)}\left(\frac{qd^{-1}Y_1^k}{qd^{-1}Y_{k+1}^{k+l}}\right)\end{array}\right]\cdot$$

$$\frac{\prod\limits_{1\le i,j\le k+l}(qd^{-1}y_i-y_j)^2}{\prod\limits_{1\le i<j\le k+l}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}\cdot C_{k+l}$$

$$\Upsilon_{2,1}(f\star g)=Sym\left[\begin{array}{l}\Upsilon(f(Y_1^k))\cdot\Upsilon(g(Y_{k+1}^{k+l}))\cdot \\ \prod\limits_{1\le i\le k}^{k<j\le k+l}\frac{(y_i-q^{-2}y_j)(y_i-q^2d^{-2}y_j)(y_i-d^2y_j)}{(y_i-y_j)^3}\end{array}\right]\cdot\frac{C_{k+l}}{C_kC_l}\cdot(qd^{-1})^{2kl}$$

This time, we take $C_k=(qd^{-1})^{-k^2}$ then $\frac{C_{k+l}}{C_kC_l}\cdot(qd^{-1})^{2kl}=1$. By picking

$$G(Y_1^k)=(qd^{-1})^{-k^2}\cdot\frac{\prod\limits_{1\le i,j\le k}(qd^{-1}y_i-y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}$$

with the parameters

$$\widetilde{q}_1=d^2q^{-2},\widetilde{q}_2=q^2,\widetilde{q}_3=d^{-2}$$

we construct an algebra homomorphism,

$$\Upsilon_{2,1}:\bigoplus_k Sh_{\widehat{\mathfrak{gl}}_2}(k,k)|_{q_1,q_2,q_3}\to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1=q_1^2,\widetilde{q}_2=q_2,\widetilde{q}_3=q_3q_1^{-1}}\qquad\text{given by}$$

$$\Upsilon_{2,1}:f\left(X_1^k|Y_1^k\right)\to f\left(qd^{-1}Y_1^k|Y_1^k\right)\cdot\frac{(qd^{-1})^{-k^2}\cdot\prod\limits_{1\le i,j\le k}(qd^{-1}y_i-y_j)^2}{\prod\limits_{1\le i<j\le k}(y_i-y_j)^2(y_i-q^2y_j)(y_i-q^{-2}y_j)}$$

As explained in the beginning, there are only 4 possible choices for λ. The above are the cases that with $\lambda=qd$ in case 1 and $\lambda=qd^{-1}$ in case 2. To get to the remaining two cases, we just need swap the roles of X_1^k and Y_1^k. We leave out the details and simply state the results:

3.6.2 Case 3 and Case 4

For $\lambda=q^{-1}d$:

$$\Upsilon_{2,1}:\bigoplus_k Sh_{\widehat{\mathfrak{gl}}_2}(k,k)|_{q_1,q_2,q_3}\to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1={q_1}^{-2},\widetilde{q}_2={q_2}^{-1},\widetilde{q}_3=q_1{q_3}^{-1}}\quad\text{given by}$$

$$\Upsilon_{2,1}:f\left(X_1^k|Y_1^k\right)\to f\left(X_1^k|qd^{-1}X_1^k\right)\cdot G(X_1^k)$$

For $\lambda = q^{-1}d^{-1}$:

$$\Upsilon_{2,1} : \bigoplus_k Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,k)|_{q_1,q_2,q_3} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}(k)|_{\widetilde{q}_1=q_3q_1{}^{-1},\widetilde{q}_2=q_2{}^{-1},\widetilde{q}_3=q_3{}^{-2}} \quad \text{given by}$$
$$\Upsilon_{2,1} : f\left(X_1^k|Y_1^k\right) \to f\left(X_1^k|qdX_1^k\right) \cdot G(X_1^k)$$

3.6.3 Summary of first example

Here, we shall point out that there are no other specializations because we have to map both X_1^k and Y_1^k to the same color, and we know that the quotient between their images must be our choices of λ, namely $q_1^{\pm 1}, q_3^{\pm 1}$ according to the wheel conditions. The last thing we need to check is whether $\Upsilon(f)$ satisfies the wheel conditions of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$.This is clear as we examine the variables in Y_1^k which is showed in figure 5. For example, as to case 1, where $\lambda = qd$. If $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}$ wheel in color $(0,0,1)$, then there exist x_a, y_b and x_c, such that $y_b = qdx_a$ and $x_c = qd^{-1}y_b$. Since $x_i = qdy_i$, let's assume $y_a = t$, then $x_a = qdt, y_b = q^2d^2t, y_c = q^2t$, therefore, $y_b = \widetilde{qd}y_a, y_c = \widetilde{qd^{-1}}y_b$, which is exactly the wheel conditions for $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}$.

Here, we have established the simplest case of $n = 2$ of Theorem 3.2.

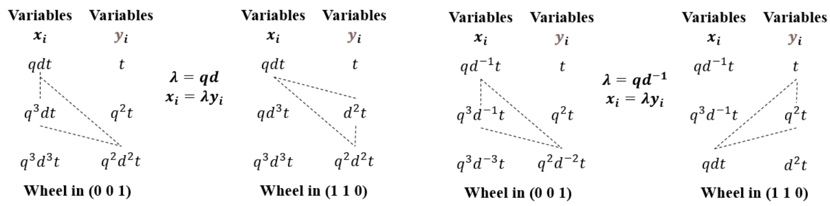

Figure 5: Check of wheel conditions

3.7 From $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ to $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}$

We utilize the same idea as explained in the section 3.4 to construct the homomorphism $\Upsilon_{3,2} : \ Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}(k,k,n) \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,n)$. In $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}(k,k,n)$, we have variables of $X_{1,1}^{1,k}, X_{2,1}^{2,k}$ and $X_{3,1}^{3,n}$ in three different colors. To simplify notations, we rename $X_{1,1}^{1,k}, X_{2,1}^{2,k}$ and $X_{3,1}^{3,n}$ by X_1^k, Y_1^k and Z_1^n respectively and set X_1^k as color 0, Y_1^k as color 1 and Z_1^n as color 2. Since X_1^k and Y_1^k have the same dimension, according to the analysis in section 3.4 on constructing algebra homomorphism from $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,k)$ to $Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}(k)$ and the shuffle factor of $\omega_{i,j}^{(3)}$, we specialize Y_1^k to λX_1^k with $\lambda \in \left\{q_1^{-1}, q_3\right\}$ and Z_1^n to μZ_1^n. Therefore, the map $\Upsilon_{3,2}$ is of the following form:

$$\Upsilon_{3,2} : f\left(X_1^k|Y_1^k|Z_1^n\right) \to f\left(X_1^k|\lambda X_1^k|\mu Z_1^n\right) \cdot G\left(X_1^k|Z_1^n\right)$$

Our goal is to determine the rational factor $G\left(X_1^k|Z_1^n\right)$, as well as λ and μ in terms of q, d, and get the explicit relation between q_1, q_2, q_3 and $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3$ so that the map of $\Upsilon_{3,2}$ is compatible with the shuffle product. By utilizing the wheel conditions, we find that $\lambda \in \left\{q^{-1}d^{-1}, qd^{-1}\right\}$.

3.7.1 case 1: $\lambda = q_3 = q^{-1}d^{-1}$

We are looking for a map of the following form:

$$\Upsilon_{3,2} : f\left(X_1^k|Y_1^k|Z_1^n\right) \to f\left(X_1^k|q^{-1}d^{-1}X_1^k|\mu Z_1^n\right) \cdot G\left(X_1^k|Z_1^n\right)$$

In order to find the rational factor of $G\left(X_1^k|Z_1^n\right)$ that ensure $\Upsilon_{3,2}$ is an algebra homomorphism, we need to analyze the conditions that G needs to meet.The numerator of G should be able to clear off the denominator of $f\left(X_1^k|q^{-1}d^{-1}X_1^k|\mu Z_1^n\right)$. In addition, the denominator of G should create the desired pole involving $\prod\limits_{1\le i\le k}^{1\le j\le n}(x_i - z_j)$ in $Sh_{\widehat{\mathfrak{gl}_2}}(k,n)$. Finally, it should clear off zeroes due to the extra wheel conditions of $Sh_{\widehat{\mathfrak{gl}_3}}(k,k,n)$. Based on the above analysis, we have:

$$G\left(X_1^k|Z_1^n\right) = \frac{\prod\limits_{1\le i,j\le k}(x_i - q^{-1}d^{-1}x_j)\prod\limits_{1\le i\le k}^{1\le j\le n}(q^{-1}d^{-1}x_i - \mu z_j)(\mu z_j - x_i)}{\prod\limits_{1\le i\le k}^{1\le j\le n}(x_i - z_j)^2\prod\limits_{1\le i<j\le k}(x_i - q^2x_j)(x_i - q^{-2}x_j)} \cdot C_{k,n}$$

Let $f\left(X_1^k|Y_1^k|Z_1^n\right) \in Sh_{\widehat{\mathfrak{gl}_3}}(k,k,n)$ and $g\left(X_{k+1}^{k+l}|Y_{k+1}^{k+l}|Z_{n+1}^{n+m}\right) \in Sh_{\widehat{\mathfrak{gl}_3}}(l,l,m)$. The choice of λ, μ and G should ensure that the equation(14) holds.

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right) \cdot \frac{G\left(X_1^{k+l}|Z_1^{n+m}\right)}{G\left(X_1^k|Z_1^n\right)G\left(X_{k+1}^{k+l}|Z_{n+1}^{n+m}\right)} = \omega^{(2)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right) \tag{14}$$

Let's compute each of the factors in the equation (14)

$$\begin{aligned}\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right) = {} & \omega_{i,i}^{(3)}\left(\frac{X_1^k}{X_{k+1}^{k+l}}\right) \cdot \omega_{i,i}^{(3)}\left(\frac{\lambda X_1^k}{\lambda X_{k+1}^{k+l}}\right) \cdot \omega_{i,i}^{(3)}\left(\frac{\mu Z_1^n}{\mu Z_{n+1}^{n+m}}\right) \cdot \\ & \omega_{i,i+1}^{(3)}\left(\frac{X_1^k}{\lambda X_{k+1}^{k+l}}\right) \cdot \omega_{i,i+1}^{(3)}\left(\frac{\lambda X_1^k}{\mu Z_{n+1}^{n+m}}\right) \cdot \omega_{i,i+1}^{(3)}\left(\frac{\mu Z_1^n}{X_{k+1}^{k+l}}\right) \cdot \\ & \omega_{i,i-1}^{(3)}\left(\frac{X_1^k}{\mu Z_{n+1}^{n+m}}\right) \cdot \omega_{i,i-1}^{(3)}\left(\frac{\lambda X_1^k}{X_{k+1}^{k+l}}\right) \cdot \omega_{i,i-1}^{(3)}\left(\frac{\mu Z_1^n}{\lambda X_{k+1}^{k+l}}\right)\end{aligned}$$

That is,

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right)=\prod_{1\le i\le k}^{n<r\le n+m}\left(\frac{x_i-qd^{-1}\mu z_r}{x_i-\mu z_r}\cdot\frac{d^{-2}q^{-1}x_i-q\mu z_r}{d^{-1}q^{-1}x_i-\mu z_r}\right)\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{x_i-q^{-2}x_j}{x_i-x_j}\cdot\frac{d^{-1}(x_i-x_j)}{x_i-d^{-1}q^{-1}x_j}\cdot\frac{x_i-q^{-2}x_j}{x_i-x_j}\cdot\frac{d^{-1}q^{-1}x_i-qd^{-1}x_j}{d^{-1}q^{-1}x_i-x_j}\right)\cdot$$

$$\prod_{1\le r\le n}^{n<s\le n+m}\frac{z_r-q^{-2}z_s}{z_r-z_s}\cdot\prod_{1\le r\le n}^{k<j\le k+l}\left(\frac{d^{-1}\mu z_r-qx_j}{\mu z_r-x_j}\cdot\frac{\mu z_r-d^{-2}x_j}{\mu z_r-d^{-1}q^{-1}x_j}\right)\quad(15)$$

The rational factor of G is,

$$\frac{G\left(X_1^{k+l}|\mu Z_1^{n+m}\right)}{G\left(X_1^k|\mu Z_1^n\right)G\left(X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}\right)}=\frac{C_{k+l,n+m}}{C_{k,n}C_{l,m}}\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\frac{(x_i-d^{-1}q^{-1}x_j)(x_j-d^{-1}q^{-1}x_i)}{(x_i-q^2x_j)(x_i-q^{-2}x_j)}\cdot\prod_{1\le i\le k}^{n<r\le n+m}\frac{(d^{-1}q^{-1}x_i-\mu z_r)(\mu z_r-x_i)}{(x_i-z_r)^2}\cdot$$

$$\prod_{1\le r\le n}^{k<j\le k+l}\frac{(d^{-1}q^{-1}x_j-\mu z_r)(\mu z_r-x_j)}{(x_j-z_r)^2}\quad(16)$$

The RHS of the equation (14) is :

$$\prod_{1\le i\le k}^{k<j\le k+l}\frac{x_i-\widetilde{q}^{-2}x_j}{x_i-x_j}\cdot\prod_{1\le r\le n}^{n<s\le n+m}\frac{z_r-\widetilde{q}^{-2}z_s}{z_r-z_s}\cdot$$

$$\prod_{1\le i\le k}^{n<r\le n+m}\frac{(x_i-\widetilde{d}^{-1}\widetilde{q}z_r)(x_i-\widetilde{d}\widetilde{q}z_r)}{(x_i-z_r)^2}\cdot\prod_{k<j\le k+l}^{1\le r\le n}\frac{(z_r-\widetilde{d}^{-1}\widetilde{q}x_j)(z_r-\widetilde{d}\widetilde{q}x_j)}{(z_r-x_j)^2}\quad(17)$$

Comparing terms of (z_r,z_s) in the equation of (15),(16) and (17) we get $\widetilde{q}=q$. Comparing terms of (x_i,z_r) with $1\le i\le k, n<r\le n+m$, in the equation of (15),(16) and (17), we get condition:

$$\{\widetilde{q}\widetilde{d}^{-1},\widetilde{q}\widetilde{d}\}=\{qd^{-1}\mu,q^2d^2\mu\}\quad(18)$$

Comparing terms (x_j,z_r) with $k<j\le k+l, 1\le r<n$, in the equation of (15),(16) and (17) we get condition:

$$\{\widetilde{q}\widetilde{d}^{-1},\widetilde{q}\widetilde{d}\}=\{qd\mu^{-1},d^{-2}\mu^{-1}\}\quad(19)$$

As far as condition (18) is concerned,

$$\widetilde{q}\widetilde{d}^{-1}\cdot\widetilde{q}\widetilde{d}=qd^{-1}\mu\cdot q^2d^2\mu\Rightarrow\widetilde{q}^2=q^3d\mu^2\Rightarrow\mu=(qd)^{-1/2}$$

When $\mu=(qd)^{-1/2}$, we have $qd\mu^{-1}=(qd)^{3/2}=q^2d^2\mu$ and $d^{-2}\mu^{-1}=q^{1/2}d^{-3/2}=qd^{-1}\mu$. Therefore, (18) and (19) are compatible, and we also have $\{\widetilde{d}^{\pm1}\}=\{(q^{1/2}d^{3/2})^{\pm1}\}$.
The remaining thing to be verified is that the image satisfies wheel condition of $Sh_{\widehat{\mathfrak{gl}}_2}(k,n)$.

1. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(0,0,2)$ implies vanishing as $\frac{x_a}{\mu z_c} = \frac{q}{d}, \frac{\mu z_c}{x_b} = qd$ for some $1 \leq a, b \leq k, 1 \leq c \leq n$. Now $\mu = (qd)^{-1/2}$, we have $\frac{x_a}{z_c} = q^{\frac{1}{2}} d^{-\frac{3}{2}} = \widetilde{q}\widetilde{d}^{-1}$ and $\frac{z_c}{x_b} = q^{\frac{3}{2}} d^{\frac{3}{2}} = \widetilde{q}\widetilde{d}$ which is exactly one of the wheel conditions for $Sh_{\widehat{\mathfrak{gl}}_2}$.

2. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(1,1,2)$implies vanishing as $\frac{q^{-1}d^{-1}x_a}{\mu z_c} = qd$, and $\frac{\mu z_c}{q^{-1}d^{-1}x_b} = \frac{q}{d}$. We have $\frac{x_a}{z_c} = q^{\frac{3}{2}} d^{\frac{3}{2}} = \widetilde{q}\widetilde{d}$, $\frac{z_c}{x_b} = q^{\frac{1}{2}} d^{-\frac{3}{2}} = \widetilde{q}\widetilde{d}^{-1}$ which is exactly one of the wheel conditions for $Sh_{\widehat{\mathfrak{gl}}_2}$.

3. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in $(2,2,0)$ and $(2,2,1)$ are checked in the same way.

A direct computation shows that the constant c_k, n can be nicely incorporated into G, by replacing G with $\tilde{G}$:

$$\widetilde{G}\left(X_1^k | Z_1^n\right) = \frac{\prod\limits_{1\leq i,j\leq k} (dx_i - d^2 q x_j)}{\prod\limits_{1\leq i<j\leq k} (x_i - q^2 x_j)(x_i - q^{-2} x_j)} \cdot \prod_{1\leq i\leq k}^{1\leq r\leq n} \frac{(x_i - q^{\frac{1}{2}} d^{\frac{1}{2}} z_r)(dx_i - q^{-\frac{1}{2}} d^{\frac{1}{2}} z_r)}{(x_i - z_r)^2}$$

with parameters $\widetilde{q}_1 = q^{-\frac{1}{2}} d^{\frac{3}{2}}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q^{-\frac{3}{2}} d^{-\frac{3}{2}}$
we have constructed an algebra homomorphism:

$$\Upsilon_{3,2} : \bigoplus_{k,n} Sh_{\widehat{\mathfrak{gl}}_3}(k,k,n)|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_2}(k,n)|_{\widetilde{q}_1 = q_1 q_3^{-1/2}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3^{3/2}} \quad \text{given by}$$

$$\Upsilon_{3,2} : f\left(X_1^k | Y_1^k | Z_1^n\right) \to f\left(X_1^k | q^{-1} d^{-1} X_1^k | q^{-\frac{1}{2}} d^{-\frac{1}{2}} Z_1^n\right) \cdot \widetilde{G}\left(X_1^k | Z_1^n\right)$$

In addition to $Y_1^k = q^{-1} d^{-1} X_1^k$, we have three other specializations to consider which are $Y_1^k = q^{-1} d X_1^k$, $Y_1^k = qd X_1^k$ and $Y_1^k = q d^{-1} X_1^k$.

3.7.2 case 2: $\lambda = q_1^{-1} = qd^{-1}$

When $Y_1^k = qd^{-1} X_1^k$, we are looking for a map of the following form:

$$\Upsilon_{3,2} : f\left(X_1^k | Y_1^k | Z_1^n\right) \to f\left(X_1^k | qd^{-1} X_1^k | \mu Z_1^n\right) \cdot G\left(X_1^k | Z_1^n\right)$$

The rational factor of G is:

$$G\left(X_1^k | Z_1^n\right) = \frac{\prod\limits_{1\leq i,j\leq k} (x_i - qd^{-1} x_j) \prod\limits_{1\leq i\leq k}^{1\leq j\leq n} (qd^{-1} x_i - \mu z_j)(\mu z_j - x_i)}{\prod\limits_{1\leq i\leq k}^{1\leq j\leq n} (x_i - z_j)^2 \prod\limits_{1\leq i<j\leq k} (x_i - q^2 x_j)(x_i - q^{-2} x_j)} \cdot C_{k,n}$$

For $f\left(X_1^k|Y_1^k|Z_1^n\right) \in Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}(k,k,n)$ and $g\left(X_{k+1}^{k+l}|Y_{k+1}^{k+l}|Z_{n+1}^{n+m}\right) \in Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}(l,l,m)$

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right) =$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{x_i-q^{-2}x_j}{x_i-x_j}\cdot\frac{d^{-1}x_i-q^2d^{-1}x_j}{x_i-qd^{-1}x_j}\cdot\frac{x_i-q^{-2}x_j}{x_i-x_j}\cdot\frac{qd^{-1}x_i-qd^{-1}x_j}{qd^{-1}x_i-x_j}\right)\cdot$$
$$\prod_{1\le i\le k}^{n<s\le n+m}\left(\frac{x_i-qd^{-1}\mu z_s}{x_i-\mu z_s}\cdot\frac{qd^{-2}x_i-q\mu z_s}{qd^{-1}x_i-\mu z_s}\right)\cdot\prod_{1\le r\le n}^{n<s\le n+m}\left(\frac{z_r-q^{-2}z_s}{z_r-z_s}\right)\cdot$$
$$\prod_{1\le r\le n}^{k<j\le k+l}\left(\frac{d^{-1}\mu z_r-qx_j}{\mu z_r-x_j}\cdot\frac{\mu z_r-q^2d^{-2}x_j}{\mu z_r-qd^{-1}x_j}\right)$$

and the ratio of G is:

$$\frac{G\left(X_1^{k+l}|Z_1^{n+m}\right)}{G\left(X_1^k|Z_1^n\right)G\left(X_{k+1}^{k+l}|Z_{n+1}^{n+m}\right)} = \frac{C_{k+l,n+m}}{C_{k,n}C_{l,m}}\cdot$$
$$\prod_{1\le i\le k}^{k<j\le k+l}\frac{(x_i-qd^{-1}x_j)(x_j-qd^{-1}x_i)}{(x_i-q^2x_j)(x_i-q^{-2}x_j)}\cdot\prod_{1\le i\le k}^{n<s\le n+m}\frac{(qd^{-1}x_i-\mu z_s)(\mu z_s-x_i)}{(x_i-z_s)^2}\cdot$$
$$\prod_{1\le r\le n}^{k<j\le k+l}\frac{(qd^{-1}x_j-\mu z_r)(\mu z_r-x_j)}{(x_j-z_r)^2}$$

By taking the product of above two factors and comparing the terms of (z_r,z_s), (x_i,x_j), (x_i,z_s), and (x_j,z_r), with the corresponding item in $\omega^{(2)}\left(\frac{X_1^k|\lambda X_1^k|\mu Z_1^n}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\mu Z_{n+1}^{n+m}}\right)$, we have $\mu = q_1^{-1/2}, \widetilde{q}_1 = q_1q_1^{1/2}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3q_1^{-1/2}$. The remaining thing to be verified is that the image satisfies wheel condition of $Sh_{\widehat{\widehat{\mathfrak{gl}}}_2}(k,n)$.

1. $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ wheel in color $(0,0,2)$ implies vanishing when $\frac{x_a}{\mu z_c} = \frac{q}{d}, \frac{\mu z_c}{x_b} = qd$ which is equivalent to $\frac{x_a}{z_c} = q^{\frac{3}{2}}d^{-\frac{3}{2}}, \frac{z_c}{x_b} = q^{\frac{1}{2}}d^{\frac{3}{2}}$ which is also equivalent to $\frac{x_a}{z_c} = \widetilde{q}\widetilde{d}^{-1}, \frac{z_c}{x_b} = \widetilde{q}\widetilde{d}$.

2. $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ wheel in color $(1,1,2)$implies vanishing when $\frac{qd^{-1}x_a}{\mu z_c} = qd, \frac{\mu z_c}{qd^{-1}x_b} = \frac{q}{d}$ which is equivalent to $\frac{x_a}{z_c} = q^{\frac{1}{2}}d^{\frac{3}{2}}, \frac{z_c}{x_b} = q^{\frac{3}{2}}d^{-\frac{3}{2}}$ which is also equivalent to $\frac{x_a}{z_c} = \widetilde{q}\widetilde{d}, \frac{z_c}{x_b} = \widetilde{q}\widetilde{d}^{-1}$.

3. $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ wheel in color $(2,2,0)$implies vanishing when $\frac{\mu z_a}{x_c} = qd, \frac{x_c}{\mu z_b} = \frac{q}{d}$ which is equivalent to $\frac{z_a}{x_c} = q^{\frac{1}{2}}d^{\frac{3}{2}} = \widetilde{q}\widetilde{d}, \frac{x_c}{z_b} = q^{\frac{3}{2}}d^{-\frac{3}{2}} = \widetilde{q}\widetilde{d}^{-1}$.

4. $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ wheel in color $(2,2,1)$implies vanishing when $\frac{\mu z_a}{qd^{-1}x_c} = qd^{-1}, \frac{qd^{-1}x_c}{\mu z_b} = qd$ which is equivalent to $\frac{z_a}{x_c} = q^{\frac{3}{2}}d^{-\frac{3}{2}} = \widetilde{q}\widetilde{d}^{-1}, \frac{x_c}{z_b} = q^{\frac{1}{2}}d^{\frac{3}{2}} = \widetilde{q}\widetilde{d}$.

By straightforward computation, we can incorporate the constant $c_{k,n}$ in G by replacing G with $\tilde{G}$, and we will get algebra homomorphism:

$$\Upsilon_{3,2}:\bigoplus_{k,n} Sh_{\widehat{\mathfrak{gl}}_3}(k,k,n)|_{q_1,q_2,q_3}\to Sh_{\widehat{\mathfrak{gl}}_2}(k,n)|_{\widetilde{q}_1=q_1q_1^{1/2},\widetilde{q}_2=q_2,\widetilde{q}_3=q_3q_1^{-1/2}} \quad \text{given by}$$

$$\Upsilon_{3,2}: f\left(X_1^k|Y_1^k|Z_1^n\right)\to f\left(X_1^k|qd^{-1}X_1^k|q^{\frac{1}{2}}d^{-\frac{1}{2}}Z_1^n\right)\cdot\widetilde{G}\left(X_1^k|Z_1^n\right) \quad \text{with}$$

$$\widetilde{G}\left(X_1^k|Z_1^n\right)=\frac{\prod\limits_{1\le i,j\le k}(x_i-qd^{-1}x_j)}{\prod\limits_{1\le i<j\le k}(x_i-q^2x_j)(x_i-q^{-2}x_j)}\cdot \prod_{1\le i\le k}^{1\le r\le n}\frac{(d^{-\frac{1}{2}}q^{\frac{1}{2}}x_i-z_r)(d^{\frac{1}{2}}q^{-\frac{1}{2}}x_i-z_r)}{(x_i-z_r)^2}$$

In summary, we have two homomorphism of $\Upsilon_{3,2}$, which are:

$$\Upsilon_{3,2}: f\left(X_1^k|Y_1^k|Z_1^n\right)\to f\left(X_1^k|q_3X_1^k|q_3^{\frac{1}{2}}Z_1^n\right)\cdot\widetilde{G}\left(X_1^k|Z_1^n\right)$$
$$\Upsilon_{3,2}: f\left(X_1^k|Y_1^k|Z_1^n\right)\to f\left(X_1^k|q_1^{-1}X_1^k|q_1^{-\frac{1}{2}}Z_1^n\right)\cdot\widetilde{G}'\left(X_1^k|Z_1^n\right)$$

3.8 From $Sh_{\widehat{\mathfrak{gl}}_3}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$

We can construct the homomorphism $\Upsilon_{3,1}$ from $Sh_{\widehat{\mathfrak{gl}}_3}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$ in two ways. One is a two-step method. That is, we firstly construct the homomorphisms $\Upsilon_{3,2}$ From $Sh_{\widehat{\mathfrak{gl}}_3}$ to $Sh_{\widehat{\mathfrak{gl}}_2}$, then construct the homomorphisms $\Upsilon_{2,1}$ from $Sh_{\widehat{\mathfrak{gl}}_2}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$, and finally compose two homomorphisms $\Upsilon_{3,2}$ and $\Upsilon_{2,1}$ together. The other way is a one-step method, which is similar to what we have done in constructing the homomorphism from $Sh_{\widehat{\mathfrak{gl}}_2}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$.
In this section, we chose the latter. We specify X_1^k color 0, Y_1^k color 1 and Z_1^k color 2. We propose the following:

Proposition 3.9 *There are four ways to map $Sh_{\widehat{\mathfrak{gl}}_3}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$ via a component-wise in the general form: $\Upsilon_{3,1}: f\left(X_1^k|Y_1^k|Z_1^k\right)\to f\left(X_1^k|\lambda X_1^k|\lambda\mu X_1^k\right)\cdot G\left(X_1^k\right)$ for $\lambda,\mu\in\{q_3,q_1^{-1}\}$.*

We will provide the more specific constructions of the four mappings in the case studies bellow:

3.9.1 Case 1: $\lambda=\mu=q_3=q^{-1}d^{-1}$

In this case, we specialize $Y_1^k=q^{-1}d^{-1}X_1^k$, and $Z_1^k=q^{-2}d^{-2}X_1^k$
The factor of $G\left(X_1^k\right)$ should meet following conditions:
1. The numerator of $G\left(X_1^k\right)$ should be able to clear off the denominator of $f\left(X_1^k|\lambda X_1^k|\lambda\mu X_1^k\right)$;

2. The denominator of $G\left(X_1^k\right)$ should create the desired pole in $Sh_{\widehat{\mathfrak{gl}}_1}$, and clear off zeroes due to wheel conditions. Based on the above analysis, we have:

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\leq i,j\leq k} (x_i - q^{-1}d^{-1}x_j)(q^{-1}d^{-1}x_i - q^{-2}d^{-2}x_j)(q^{-2}d^{-2}x_i - x_j)}{\prod\limits_{1\leq i<j\leq k} (x_i - x_j)^2 (x_i - q^2 x_j)^2 (x_i - q^{-2}x_j)^2} \cdot C_k$$

In order to determine the explicit relation between q_1, q_2, q_3 and $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3$, let $f\left(X_1^k|Y_1^k|Z_1^k\right) \in Sh_{\widehat{\mathfrak{gl}}_3}(k,k,k)$ and $g\left(X_{k+1}^{k+l}|Y_{k+1}^{k+l}|Z_{k+1}^{k+l}\right) \in Sh_{\widehat{\mathfrak{gl}}_3}(l,l,l)$.
We write down the shuffle factors:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right) =$$
$$\prod_{1\leq i\leq k}^{k<j\leq k+l}\left(\frac{x_i - q^{-2}x_j}{x_i - x_j}\cdot\frac{x_i - q^{-2}x_j}{x_i - x_j}\cdot\frac{x_i - q^{-2}x_j}{x_i - x_j}\right)\cdot$$
$$\prod_{1\leq i\leq k}^{k<j\leq k+l}\left(\frac{d^{-1}(x_i - x_j)}{x_i - q^{-1}d^{-1}x_j}\cdot\frac{x_i - q^{-1}d^{-3}x_j}{x_i - q^{-2}d^{-2}x_j}\cdot\frac{q^{-1}d^{-2}x_i - q^{-1}d^{-2}x_j}{q^{-1}d^{-1}x_i - q^{-2}d^{-2}x_j}\right)\cdot$$
$$\prod_{1\leq i\leq k}^{k<j\leq k+l}\left(\frac{q^{-1}d^{-1}x_i - qd^{-1}x_j}{q^{-1}d^{-1}x_i - x_j}\cdot\frac{q^{-2}d^{-3}x_i - qx_j}{q^{-2}d^{-2}x_i - x_j}\cdot\frac{q^{-2}d^{-2}x_i - d^{-2}x_j}{q^{-2}d^{-2}x_i - q^{-1}d^{-1}x_j}\right)$$

The ratio of $\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)}$, is equal to :

$$\prod_{1\leq i\leq k}^{k<j\leq k+l}\frac{(x_i - q^{-1}d^{-1}x_j)(q^{-1}d^{-1}x_i - q^{-2}d^{-2}x_j)(q^{-2}d^{-2}x_i - x_j)}{(x_i - x_j)^2}\cdot$$
$$\prod_{1\leq i\leq k}^{k<j\leq k+l}\left(\frac{(x_j - q^{-1}d^{-1}x_i)(q^{-1}d^{-1}x_j - q^{-2}d^{-2}x_i)(q^{-2}d^{-2}x_j - x_i)}{(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}\right)\cdot\frac{G_{k+l}}{G_kG_l}$$

The product of the above two factors is:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right)\cdot\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)} =$$
$$\frac{-1}{q^6d^{10}}\cdot\prod_{1\leq i\leq k}^{k<j\leq k+l}\frac{(x_i - q^{-2}x_j)(x_i - q^{-1}d^{-3}x_j)(x_i - q^3d^3x_j)}{(x_i - x_j)^3}$$

We need to verify that the image satisfies wheel condition of $Sh_{\widehat{\mathfrak{gl}}_1}$.

1. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(0,0,2)$ implies vanishing as $\frac{x_a}{q^{-2}d^{-2}x_b} = \frac{q}{d}, \frac{q^{-2}d^{-2}x_b}{x_c} = qd$ for some $1 \leq a,b,c \leq k$ which means $\frac{x_a}{x_b} = \frac{1}{qd^3}, \frac{x_b}{x_c} = q^3d^3$ is equal to $\frac{x_a}{x_b} = \widetilde{q}_1^{-1}, \frac{x_b}{x_c} = \widetilde{q}_3^{-1}$.

2. The $Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}$ wheel in color $(0,2,2)$ implies vanishing as $\frac{q^{-2}d^{-2}x_a}{x_b} = qd$, $\frac{x_b}{q^{-2}d^{-2}x_c} = qd^{-1}$ which means $\frac{x_a}{x_b} = \frac{1}{q^3d^3}$, $\frac{x_b}{x_c} = q^{-1}d^{-3}$ is equal to $\frac{x_a}{x_b} = \widetilde{q}_3^{-1}$, $\frac{x_b}{x_c} = \widetilde{q}_1^{-1}$.

3. Other cases can be verified similarly.

Thus, we have constructed a homomorphism $\Upsilon_{3,1}$

$$\Upsilon_{3,1} : \bigoplus_k Sh_{\widehat{\widehat{\mathfrak{gl}}}_3}(k,k,k)|_{q_1,q_2,q_3} \to Sh_{\widehat{\widehat{\mathfrak{gl}}}_1}(k)|_{\widetilde{q}_1=q_1q_3^{-2},\widetilde{q}_2=q_2,\widetilde{q}_3=q_3^3} \qquad \text{given by}$$

$$\Upsilon_{3,1} : f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(X_1^k|q^{-1}d^{-1}X_1^k|q^{-2}d^{-2}X_1^k\right) \cdot G\left(X_1^k\right) \qquad \text{with}$$

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\le i,j\le k} (x_i - q^{-1}d^{-1}x_j)(q^{-1}d^{-1}x_i - q^{-2}d^{-2}x_j)}{\prod\limits_{1\le i<j\le k} (x_i - x_j)^2(x_i - q^2x_j)^2} \cdot \frac{\prod\limits_{1\le i,j\le k} (q^{-2}d^{-2}x_i - x_j)}{\prod\limits_{1\le i<j\le k} (x_i - q^{-2}x_j)^2} \cdot (-q^3d^5)^k$$

The existence of such constant c_k is due to the criteria of Remark 3.6.

3.9.2 Case 2: $\lambda = \mu = q_1^{-1} = qd^{-1}$

In this case, we specialize $Y_1^k = qd^{-1}X_1^k$ and $Z_1^k = q^2d^{-2}X_1^k$
The $G\left(X_1^k\right)$ should be of the form:

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\le i,j\le k} (x_i - qd^{-1}x_j)(qd^{-1}x_i - q^2d^{-2}x_j)(q^2d^{-2}x_i - x_j)}{\prod\limits_{1\le i<j\le k} (x_i - x_j)^2(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2} \cdot C_k$$

We write down the shuffle factors:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right) = \prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{x_i - q^{-2}x_j}{x_i - x_j}\right)^3 \cdot$$
$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{qd^{-2}x_i - q^3d^{-2}x_j}{qd^{-1}x_i - q^2d^{-2}x_j} \cdot \frac{q^2d^{-3}x_i - qx_j}{q^2d^{-2}x_i - x_j} \cdot \frac{x_i - q^3d^{-3}x_j}{x_i - q^2d^{-2}x_j}\right) \cdot$$
$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{qd^{-1}x_i - qd^{-1}x_j}{qd^{-1}x_i - x_j} \cdot \frac{q^2d^{-2}x_i - q^2d^{-2}x_j}{q^2d^{-2}x_i - qd^{-1}x_j} \cdot \frac{d^{-1}x_i - q^2d^{-1}x_j}{x_i - qd^{-1}x_j}\right)$$

The ratio of $\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)}$ is equal to :

$$\prod_{\substack{1\le i\le k}}^{k<j\le k+l}\left(\frac{(x_i-qd^{-1}x_j)(qd^{-1}x_i-q^2d^{-2}x_j)(q^2d^{-2}x_i-x_j)}{(x_i-x_j)^2}\right)\cdot$$
$$\prod_{\substack{1\le i\le k}}^{k<j\le k+l}\left(\frac{(x_j-qd^{-1}x_i)(qd^{-1}x_j-q^2d^{-2}x_i)(q^2d^{-2}x_j-x_i)}{(x_i-q^2x_j)^2(x_i-q^{-2}x_j)^2}\right)\cdot\frac{G_{k+l}}{G_kG_l}$$

The product of the above two factors is:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right)\cdot\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)}=$$
$$\prod_{\substack{1\le i\le k}}^{k<j\le k+l}\frac{(x_i-q^{-2}x_j)(x_i-q^{-1}d^3x_j)(x_i-q^3d^{-3}x_j)}{(x_i-x_j)^3}\cdot\frac{G_{k+l}}{G_kG_l}$$

The last thing to be verified is that the image satisfies wheel condition of $Sh_{\widehat{\mathfrak{gl}}_1}$.

1. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(0,0,2)$ implies vanishing as $\frac{x_a}{q^2d^{-2}x_b}=\frac{q}{d},\frac{q^2d^{-2}x_b}{x_c}=qd$ for some $1\le a,b,c\le k$, which means $\frac{x_a}{x_b}=\frac{q^3}{d^3}=\widetilde{q}_3^{-1}$, $\frac{x_b}{x_c}=q^{-1}d^3=\widetilde{q}_1^{-1}$.

2. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(2,2,0)$ implies vanishing as $\frac{q^2d^{-2}x_a}{x_b}=qd$, $\frac{x_b}{q^2d^{-2}x_c}=qd^{-1}$ which means $\frac{x_a}{x_b}=\frac{d^3}{q}=\widetilde{q}_1^{-1}$, $\frac{x_b}{x_c}=q^3d^{-3}=\widetilde{q}_3^{-1}$.

Thus, we have constructed an algebra homomorphism

$$\Upsilon_{3,1}:\bigoplus_k Sh_{\widehat{\mathfrak{gl}}_3}(k,k,k)\to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1=q_1^3,\widetilde{q}_2=q_2,\widetilde{q}_3=q_3q_1^{-2}}\qquad\text{given by}$$
$$\Upsilon_{3,1}:f\left(X_1^k|Y_1^k|Z_1^k\right)\to f\left(X_1^k|qd^{-1}X_1^k|q^2d^{-2}X_1^k\right)\cdot G\left(X_1^k\right)\qquad\text{with}$$
$$G\left(X_1^k\right)=\frac{\prod\limits_{1\le i,j\le k}(x_i-qd^{-1}x_j)^2(x_i-q^{-2}d^2x_j)}{\prod\limits_{1\le i<j\le k}(x_i-x_j)^2(x_i-q^2x_j)^2(x_i-q^{-2}x_j)^2}$$

3.9.3 Case 3: $\lambda=q_3=q^{-1}d^{-1},\mu=q_1^{-1}=qd^{-1}$

In this case, we specialize $Y_1^k=q^{-1}d^{-1}X_1^k$ and $Z_1^k=d^{-2}X_1^k$
The $G(X_1^k)$ should be of the form:

$$G(X_1^k)=\frac{\prod\limits_{1\le i,j\le k}(x_i-q^{-1}d^{-1}x_j)(q^{-1}d^{-1}x_i-d^{-2}x_j)(d^{-2}x_i-x_j)}{\prod\limits_{1\le i<j\le k}(x_i-x_j)^2(x_i-q^2x_j)^2(x_i-q^{-2}x_j)^2}\cdot C_k$$

We write down the specializations of shuffle factors:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right) = \prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{x_i - q^{-2}x_j}{x_i - x_j}\right)^3 \cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{q^{-1}d^{-2}x_i - qd^{-2}x_j}{q^{-1}d^{-1}x_i - d^{-2}x_j}\cdot\frac{d^{-3}x_i - qx_j}{d^{-2}x_i - x_j}\cdot\frac{x_i - qd^{-3}x_j}{x_i - d^{-2}x_j}\right)\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{d^{-1}x_i - d^{-1}x_j}{x_i - q^{-1}d^{-1}x_j}\cdot\frac{q^{-1}d^{-1}x_i - qd^{-1}x_j}{q^{-1}d^{-1}x_i - x_j}\cdot\frac{d^{-2}x_i - d^{-2}x_j}{d^{-2}x_i - q^{-1}d^{-1}x_j}\right)$$

The ratio of $\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)}$ is equal to :

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{(x_i - q^{-1}d^{-1}x_j)(q^{-1}d^{-1}x_i - d^{-2}x_j)(d^{-2}x_i - x_j)}{(x_i - x_j)^2}\right)\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{(x_j - q^{-1}d^{-1}x_i)(q^{-1}d^{-1}x_j - d^{-2}x_i)(d^{-2}x_j - x_i)}{(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}\right)\cdot\frac{G_{k+l}}{G_kG_l}$$

The product of the above two factors is:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right)\cdot\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)} =$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\frac{(x_i - q^{-2}x_j)(x_i - qd^3x_j)(x_i - qd^{-3}x_j)}{(x_i - x_j)^3}$$

The last thing to be verified is that the image satisfies wheel condition of $Sh_{\widehat{\mathfrak{gl}}_1}$.

1. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(0,0,2)$ implies vanishing as $\frac{x_a}{d^{-2}x_b} = \frac{q}{d}, \frac{d^{-2}x_b}{x_c} = qd$ for some $1 \le a,b,c \le k$, which means $\frac{x_a}{x_b} = qd^{-3} = \widetilde{q}_1^{-1}, \frac{x_b}{x_c} = qd^3 = \widetilde{q}_3^{-1}$.
2. The $Sh_{\widehat{\mathfrak{gl}}_3}$ wheel in color $(2,2,0)$ implies vanishing as $\frac{d^{-2}x_a}{x_b} = qd, \frac{x_b}{d^{-2}x_c} = qd^{-1}$, which means $\frac{x_a}{x_b} = qd^3 = \widetilde{q}_3^{-1}, \frac{x_b}{x_c} = qd^{-3} = \widetilde{q}_1^{-1}$.

Thus, we have constructed a possible algebra homomorphism

$$\Upsilon_{3,1} : \bigoplus_k Sh_{\widehat{\mathfrak{gl}}_3}(k,k,k)|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1=q_1^2q_3^{-1},\widetilde{q}_2=q_2,\widetilde{q}_3=q_3^2q_1^{-1}} \quad \text{given by}$$

$$\Upsilon_{3,1} : f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(X_1^k|q^{-1}d^{-1}X_1^k|d^{-2}X_1^k\right)\cdot G\left(X_1^k\right) \quad \text{with}$$

$$G\left(X_1^k\right) = \frac{\prod_{1\le i,j\le k}(x_i - q^{-1}d^{-1}x_j)(x_i - qd^{-1}x_j)(x_i - d^2x_j)}{\prod_{1\le i<j\le k}(x_i - x_j)^2(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}$$

3.9.4 Case 4: $\lambda = q_1^{-1} = qd^{-1}, \mu = q_3 = q^{-1}d^{-1}$

In this case, we specialize $Y_1^k = qd^{-1}X_1^k$ and $Z_1^k = d^{-2}X_1^k$
The $G\left(X_1^k\right)$ should be of the form:

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\le i,j\le k}(x_i - qd^{-1}x_j)(qd^{-1}x_i - d^{-2}x_j)(d^{-2}x_i - x_j)}{\prod\limits_{1\le i<j\le k}(x_i - x_j)^2(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}\cdot C_k$$

We write down the specializations of shuffle factors:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right) = \prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{x_i - q^{-2}x_j}{x_i - x_j}\right)^3\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{d^{-1}x_i - q^2d^{-1}x_j}{x_i - qd^{-1}x_j}\cdot\frac{qd^{-2}x_i - qd^{-2}x_j}{qd^{-1}x_i - d^{-2}x_j}\cdot\frac{d^{-3}x_i - qx_j}{d^{-2}x_i - x_j}\cdot\frac{x_i - qd^{-3}x_j}{x_i - d^{-2}x_j}\right)\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{qd^{-1}x_i - qd^{-1}x_j}{qd^{-1}x_i - x_j}\cdot\frac{d^{-2}x_i - q^2d^{-2}x_j}{qd^{-2}x_i - qd^{-1}x_j}\right)$$

The ratio of $\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)}$ is equal to :

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{(x_i - qd^{-1}x_j)(qd^{-1}x_i - d^{-2}x_j)(d^{-2}x_i - x_j)}{(x_i - x_j)^2}\right)\cdot$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\left(\frac{(x_j - qd^{-1}x_i)(qd^{-1}x_j - d^{-2}x_i)(d^{-2}x_j - x_i)}{(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}\right)\cdot\frac{G_{k+l}}{G_kG_l}$$

The product of the above two factors is:

$$\omega^{(3)}\left(\frac{X_1^k|\lambda X_1^k|\lambda\mu X_1^k}{X_{k+1}^{k+l}|\lambda X_{k+1}^{k+l}|\lambda\mu X_{k+1}^{k+l}}\right)\cdot\frac{G(X_1^{k+l})}{G(X_1^k)G(X_1^l)} =$$

$$\prod_{1\le i\le k}^{k<j\le k+l}\frac{(x_i - q^{-2}x_j)(x_i - qd^3x_j)(x_i - qd^{-3}x_j)}{(x_i - x_j)^3}$$

The last thing to be verified is that the image satisfies wheel condition of $Sh_{\widehat{\mathfrak{gl}}_1}$.
Since $\widetilde{q}_1, \widetilde{q}_2$ and $\widetilde{q}_3$ is equal to those in case 3, they also satisfy the wheel conditions. Thus, we have constructed a possible algebra homomorphism

$$\Upsilon_{3,1} : \bigoplus_k Sh_{\widehat{\mathfrak{gl}}_3}(k,k,k) \to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1 = q_1^2q_3^{-1}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3^2q_1^{-1}} \qquad \text{given by}$$

$$\Upsilon_{3,1} : f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(X_1^k|qd^{-1}X_1^k|d^{-2}X_1^k\right)\cdot G\left(X_1^k\right) \qquad \text{with}$$

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\le i,j\le k}(x_i - qd^{-1}x_j)(qd^{-1}x_i - d^{-2}x_j)(d^{-2}x_i - x_j)}{\prod\limits_{1\le i<j\le k}(x_i - x_j)^2(x_i - q^2x_j)^2(x_i - q^{-2}x_j)^2}$$

In summary, we have constructed 4 homomorphisms of the following form:

$$\Upsilon_{3,1} : \bigoplus_k Sh_{\widehat{\mathfrak{gl}}_3}(k,k,k)|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\widetilde{q}_1,\widetilde{q}_2,\widetilde{q}_3}$$
$$\Upsilon_{3,1} : f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(X_1^k|\lambda X_1^k|\lambda\mu X_1^k\right)\cdot G\left(X_1^k\right)$$

with the following values of λ,μ and $\widetilde{q}_1,\widetilde{q}_2,\widetilde{q}_3$:

1.	$\lambda = q^{-1}d^{-1}, \mu = q^{-1}d^{-1}$	$\widetilde{q}_1 = q_1 q_3^{-2}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3^3$	$\widetilde{q} = q, \widetilde{d} = q^2 d^3$
2.	$\lambda = qd^{-1}, \mu = qd^{-1}$	$\widetilde{q}_1 = q_1^3, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_1^{-2}q_3$	$\widetilde{q} = q, \widetilde{d} = q^{-2}d^3$
3.	$\lambda = q^{-1}d^{-1}, \mu = qd^{-1}$	$\widetilde{q}_1 = q_1^2 q_3^{-1}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3^3 q_1^{-1}$	$\widetilde{q} = q, \widetilde{d} = d^3$
4.	$\lambda = qd^{-1}, \mu = q^{-1}d^{-1}$	$\widetilde{q}_1 = q_1^2 q_3^{-1}, \widetilde{q}_2 = q_2, \widetilde{q}_3 = q_3^3 q_1^{-1}$	$\widetilde{q} = q, \widetilde{d} = d^3$

3.10 How many ways to map $Sh_{\widehat{\mathfrak{gl}}_3} \to Sh_{\widehat{\mathfrak{gl}}_1}$

First recall that we have 4 specializations $Sh_{\widehat{\mathfrak{gl}}_2} \to Sh_{\widehat{\mathfrak{gl}}_1}$
Next recall that we have 2 specializations $Sh_{\widehat{\mathfrak{gl}}_3} \to Sh_{\widehat{\mathfrak{gl}}_2}$. Now combining the above two specializations, we have the following possible values of $(\tilde{q},\tilde{d})$. $\tilde{q} = q$ always, and $\tilde{d} \in \{d^3q^2, d^3, d^3q^{-2}\}$

3.10.1 Case 1: $\tilde{q} = q, \tilde{d} = d^3q^2$

$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_3Z_1^k|q_3^2Z_1^k\right)\cdot G \qquad \text{or}$$
$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_3Z_1^k|q_3^{-1}Z_1^k\right)\cdot G$$

3.10.2 Case 2: $\tilde{q} = q, \tilde{d} = d^3q^{-2}$

$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_1^{-1}Z_1^k|q_1^{-2}Z_1^k\right)\cdot G \qquad \text{or}$$
$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_1^{-1}Z_1^k|q_1Z_1^k\right)\cdot G$$

3.10.3 Case 3&4: $\tilde{q} = q, \tilde{d} = d^3$

$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_3Z_1^k|q_3q_1^{-1}Z_1^k\right)\cdot G \qquad \text{or}$$
$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_3Z_1^k|q_1Z_1^k\right)\cdot G$$
$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_1^{-1}Z_1^k|q_3q_1^{-1}Z_1^k\right)\cdot G$$
$$f\left(X_1^k|Y_1^k|Z_1^k\right) \to f\left(Z_1^k|q_1^{-1}Z_1^k|q_3^{-1}Z_1^k\right)\cdot G$$

We obtain the remaining 8 from the above through cyclically rotating, which we demonstrate the notion in the graphs below. Through 2-step method, from $Sh_{\widehat{\mathfrak{gl}}_3} \to Sh_{\widehat{\mathfrak{gl}}_2} \to Sh_{\widehat{\mathfrak{gl}}_1}$ we are left with 8 specializations that can be represented by the graphs below.

They are:

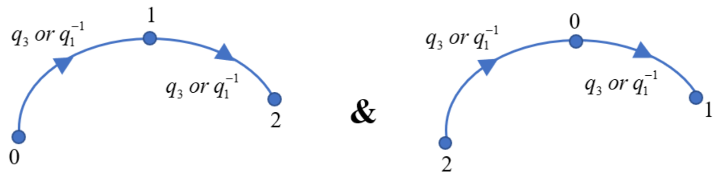

Figure 6: Symmetry of the specializations

1. $\lambda = q_3 = \frac{1}{qd}, \lambda\mu = \sqrt{\frac{1}{q}} \cdot \tilde{q}_3 = \sqrt{\frac{1}{qd}} \cdot \frac{1}{q^{1/2}d/2} = \frac{1}{q^2 d^2} = q_3^2$
2. $\lambda = q_3 = \frac{1}{qd} \quad \lambda\mu = \sqrt{\frac{1}{qd}} \cdot \widetilde{q}_1 = \sqrt{\frac{1}{q}} \cdot q_1 \cdot q_3^{-1/2} = q_1$
3. $\lambda = q_3 = \frac{1}{qd}, \quad \lambda\mu = \sqrt{\frac{1}{qd}} \cdot \widetilde{q_3^{-1}} = \sqrt{\frac{1}{qd}} \cdot q^{3/2} d^{3/2} = qd = q_3^{-1}$
4. $\lambda = q_3 = \frac{1}{qd}, \quad \lambda\mu = \sqrt{\frac{1}{qd}} \cdot \widetilde{q_1^{-1}} = \sqrt{\frac{1}{qd}} \cdot \frac{q}{d} \cdot \frac{1}{\sqrt{qd}} = \frac{1}{d^2} = q_3/q_1$
5. $\lambda = q_1{}^{-1} = \frac{q}{2}, \quad \lambda\mu = \sqrt{\frac{9}{d}} \cdot \tilde{q}_3 = \sqrt{\frac{q}{d}} \cdot \frac{1}{q^{1/2}d^{3/2}} = q^{-1}d^{-2} = q^{-1/2}q_3^{3/2}$
6. $\lambda = q_1^{-1} = \frac{q}{d}, \quad \lambda\mu = \sqrt{\frac{q}{d}} \cdot \tilde{q}_1 = \sqrt{q/d} \cdot q_1 \cdot q_3^{-1/2} = \sqrt{\frac{q}{d}} \cdot \frac{d}{q} \cdot \sqrt{qd} = d = q_1^{1/2} q_3^{-1/2}$
7. $\lambda = q^{-1} = q/d, \quad \lambda\mu = \sqrt{q/d} \cdot \tilde{q}_3^{-1} = \sqrt{q/d} \cdot q^{3/1} d^{3/2} = q^2 d = q^{-1/2} q_3^{-3/2}$
8. $\lambda = q_1^{-1} = q/d, \quad \lambda\mu = \sqrt{q/d} \cdot \widetilde{q_1^{-1}} = \sqrt{q/d} \cdot q/d \cdot \frac{1}{\sqrt{qd}} = q/d^2 = q_3^{1/2} q_1^{-3/2}$

This matches the 12 direct specializations obtained from mapping $Sh_{\widehat{\mathfrak{gl}}_3} \to Sh_{\widehat{\mathfrak{gl}}_1}$, which stemmed from three choices of initial point i, and 4 choices of q_3 vs q_1^{-1} on edges from $i \to i+1$ and $i+1 \to i+2$.
Recall that when we tried to generalize from $Sh_{\widehat{\mathfrak{gl}}_3} \to Sh_{\widehat{\mathfrak{gl}}_1}$, there were $3 \cdot 2^{3-1} = 12$ direct specialization maps.
So we expect that all composed specializations $Sh_{\widehat{\mathfrak{gl}}_n} \to Sh_{\widehat{\mathfrak{gl}}_{n_1}} \to, \ldots, \to Sh_{\widehat{\mathfrak{gl}}_{n_l}} \to Sh_{\widehat{\mathfrak{gl}}_1}$ with $1 < n_l <, \ldots, < n_1 < n$ to be the same.
Thus, we have explicitly presented 12 mappings, which is a specific case for the following, more general setting.

3.11 From $Sh_{\widehat{\mathfrak{gl}}_{n+1}}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$

Here, we have arrived at our main proposition. In order to construct more general homomorphism $Sh_{\widehat{\mathfrak{gl}}_{m+n}} \to Sh_{\widehat{\mathfrak{gl}}_n}$, we next construct the maps of $\Upsilon_{n+1,1} : Sh_{\widehat{\mathfrak{gl}}_{n+1}} \to Sh_{\widehat{\mathfrak{gl}}_1}$.

Theorem 3.12 (Second Theorem)

There are $n \cdot 2^{n-1}$ homomorphisms:

$$\Upsilon_n : \bigoplus_{k_0,k_1,\ldots,k_[n-1]} Sh_{\widehat{\mathfrak{gl}}_n}(\underbrace{k\ldots k})|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_1}(k)\,|_{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3}$$

For the special case of $m = 1$, there are $n \cdot 2^{n-1}$ many specialization maps from the imaginary part of $Sh_{\widehat{\mathfrak{gl}}_n}$ to $Sh_{\widehat{\mathfrak{gl}}_1}$. For simplicity, we let $X_{0,1}^{0,k} = X_1^k$, and $\Upsilon_{n+1,1}$ is in the form of

$$\Upsilon_{n+1,1} : f(X_{0,1}^{0,k}, X_{1,1}^{1,k}, \cdots, X_{n,1}^{n,k}) \to f(X_1^k, \lambda_1 X_1^k, \lambda_1 X_1^k, \cdots, X_1^k \prod_{i=1}^{n} \lambda_i) \cdot G$$

We know that, for $1 \leq i \leq n$, $\lambda_i \in \{q^{-1}d^{-1}, qd^{-1}\}$.We are interested in the general case, not just $\lambda_1 = \lambda_2 =, \ldots, = \lambda_n$. The $G\left(X_1^k\right)$ should be of the form:

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\leq i,j\leq k} (x_i - \lambda_1 x_j)(x_i - \lambda_2 x_j)\cdots(x_i - \lambda_n x_j)(x_i - x_j \prod\limits_{1\leq r\leq n} \lambda_r^{-1})}{\prod\limits_{1\leq i<j\leq k} (x_i - x_j)^2 (x_i - q^2 x_j)^n (x_i - q^{-2} x_j)^n} \cdot C_k$$

For $f \in Sh_{\widehat{\mathfrak{gl}}_{n+1}}(\underbrace{k,\ldots,k}_{n+1})$ and $g \in Sh_{\widehat{\mathfrak{gl}}_{n+1}}(\underbrace{l,\ldots,l}_{n+1})$, the specializations of shuffle factors $\omega^{(n+1)}\left(\frac{X_1^k,\ldots,X_1^k\prod_{i=1}^{n}\lambda_i}{X_{k+1}^{k+l},\ldots,X_{k+1}^{k+l}\prod_{i=1}^{n}\lambda_i}\right)$ is:

$$\prod_{1\leq i\leq k}^{k<j\leq k+l} \left(\frac{x_i - q^{-2}x_j}{x_i - x_j}\right)^{n+1} \cdot \frac{d^{-1}x_i - q\lambda_1 x_j}{x_i - \lambda_1 x_j} \cdot \frac{d^{-1}x_i - q\lambda_2 x_j}{x_i - \lambda_2 x_j} \cdots$$
$$\frac{d^{-1}x_i - q\lambda_n x_j}{x_i - \lambda_n x_j} \cdot \frac{d^{-1}\lambda_1\cdots\lambda_n x_i - q x_j}{\lambda_1\cdots\lambda_n x_i - x_j} \cdot \frac{\lambda_1 x_i - qd^{-1}x_j}{\lambda_1 x_i - x_j} \cdot$$
$$\frac{\lambda_2 x_i - qd^{-1}x_j}{\lambda_2 x_i - x_j} \cdots \frac{\lambda_n x_i - qd^{-1}x_j}{\lambda_n x_i - x_j} \cdot \frac{x_i - qd^{-1}\lambda_1\cdots\lambda_n x_j}{x_i - \lambda_1\cdots\lambda_n x_j}$$

The ratio of G is:

$$\prod_{1\leq i\leq k}^{k<j\leq k+l} \frac{(x_i - \lambda_1 x_j)(x_i - \lambda_2 x_j)\cdots(x_i - \lambda_n x_j)(x_i - x_j \prod\limits_{1\leq r\leq n} \lambda_r^{-1})}{(x_i - x_j)^2} \cdot$$
$$\prod_{1\leq i\leq k}^{k<j\leq k+l} \frac{(x_j - \lambda_1 x_i)(x_j - \lambda_2 x_i)\cdots(x_j - \lambda_n x_i)(x_j - x_i \prod\limits_{1\leq r\leq n} \lambda_r^{-1})}{(x_i - q^2 x_j)^n (x_i - q^{-2} x_j)^n} \cdot C_k$$

Taking the product of the above two factors,we have

$$\omega^{(n+1)}\left(\frac{X_1^k,\ldots,X_1^k\prod_{i=1}^{n}\lambda_i}{X_{k+1}^{k+l},\ldots,X_{k+1}^{k+l}\prod_{i=1}^{n}\lambda_i}\right)\cdot G\left(X_1^k\right)=\prod_{1\le i\le k}^{k<j\le k+l}\left(d^{-1}\prod_{1\le r\le n}\lambda_r\right)\cdot$$
$$\prod_{1\le i\le k}^{k\le j\le k+l}\frac{(x_i-q^{-2}x_j)(x_i-qdx_j\prod_{1\le r\le n}\lambda_r^{-1})(x_i-qd^{-1}x_j\prod_{1\le r\le n}\lambda_r^{-1})}{\left(x_i-x_j\right)^3}$$

Therefore, we have

$$\omega^{(n+1)}\left(\frac{X_1^k,\ldots,X_1^k\prod_{i=1}^{n}\lambda_i}{X_{k+1}^{k+l},\ldots,X_{k+1}^{k+l}\prod_{i=1}^{n}\lambda_i}\right)\cdot G\left(X_1^k\right)=\prod_{1\le i\le k}^{k<j\le k+l}\left(\omega^{(1)}\left(\frac{x_i}{x_j}\right)\cdot d^{-1}\prod_{1\le r\le n}\lambda_r\right)$$

$\{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3\}=\{q^2,q^{-1}d\prod_{1\le r}^{r\le n}\lambda_r^{-1},q^{-1}d^{-1}\prod_{1\le r}^{r\le n}\lambda_r\}$ that is $\tilde{q}=q,\tilde{d}=d\prod_{1\le r}^{r\le n}\lambda_r^{-1}$.

$$C_k=\prod_{1\le i\le k}^{k<j\le k+l}\left(-d^{-1}\right)^{n+1}=(-d)^{-(n+1)kl}$$

We need to verify the wheel conditions for this mapping and choice of λ:

1. The $Sh_{\widehat{\mathfrak{gl}}_{n+1}}$ wheel in colors $(n,n,0)$ implies vanishing as $\frac{x_{n,a}}{x_{0,b}}\prod_{1\le r\le n}\lambda_r=qd$ and $\frac{x_{0,b}}{x_{n,c}}\prod_{1\le r\le n}\lambda_r^{-1}=qd^{-1}$, which means $\frac{x_{n,a}}{x_{0,b}}=qd\prod_{1\le r\le n}\lambda_r^{-1}=\tilde{q}\tilde{d}=\tilde{q}_3^{-1}$, $\frac{x_{0,b}}{x_{n,c}}=qd^{-1}\prod_{1\le r\le n}\lambda_r=\tilde{q}\tilde{d}^{-1}=\tilde{q}_1^{-1}$.

2. The $Sh_{\widehat{\mathfrak{gl}}_{n+1}}$ wheel in colors $(0,0,n)$ implies vanishing as $\frac{x_{0,a}}{x_{n,b}}\prod_{1\le r\le n}\lambda_r^{-1}=qd^{-1}$ and $\frac{x_{n,b}}{x_{0,c}}\prod_{1\le r\le n}\lambda_r=qd$ which means $\frac{x_{0,a}}{x_{n,b}}=qd^{-1}\prod_{1\le r\le n}\lambda_r=\tilde{q}\tilde{d}^{-1}=\tilde{q}_1^{-1}$ and $\frac{x_{n,b}}{x_{0,c}}=qd\prod_{1\le r\le n}\lambda_r^{-1}=\tilde{q}\tilde{d}=\tilde{q}_3^{-1}$.

3. Other cases can be verified similarly.

Therefore, we have constructed a homomorphism $\Upsilon_{n+1,1}$

$$\Upsilon_{n+1,1} : Sh_{\widehat{\mathfrak{gl}}_{n+1}}(\underbrace{k,\ldots,k}_{n+1})|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_1}(k)|_{\tilde{q}_1=q_1\prod\limits_{r=1}^{n}\lambda_r^{-1},\tilde{q}_2=q_2,\tilde{q}_3=q_3\prod\limits_{r=1}^{n}\lambda_r}$$

given by

$$\Upsilon_{n+1,1} : f(X_{0,1}^{0,k}, X_{1,1}^{1,k}\cdots X_{n,1}^{n,k}) \to f(X_1^k, \lambda_1 X_1^k \cdots X_1^k \prod_{i=1}^{n}\lambda_i)\cdot G$$

with

$$G\left(X_1^k\right) = \frac{\prod\limits_{1\le i,j\le k}(x_i-\lambda_1x_j)(x_i-\lambda_2x_j)\cdots(x_i-\lambda_nx_j)(x_i-\prod\limits_{1\le r\le n}\lambda_r^{-1}x_j)}{\prod\limits_{1\le i<j\le k}(x_i-x_j)^2(x_i-q^2x_j)^n(x_i-q^{-2}x_j)^n}\cdot C_k$$

$$C_k = \prod_{1\le i\le k}^{k<j\le k+l}\left(-d^{-1}\right)^{n+1} = (-d)^{-(n+1)kl}$$

Note:

1. If $\lambda_1=\lambda_2=\ldots=\lambda_n=q_3$ then $\tilde{q}_1=q_1q_3^{-n}$, $\tilde{q}_3=q_3q_3^n$

2. If $\lambda_1=\lambda_2=\ldots=\lambda_n={q_1}^{-1}$ then $\tilde{q}_1=q_1q_1^n$, $\tilde{q}_3=q_3q_1^{-n}$

3.13 From $Sh_{\widehat{\mathfrak{gl}}_{m+n}}$ to $Sh_{\widehat{\mathfrak{gl}}_n}$

Finally, we shall establish our main result of Theorem 3.1 by providing all of the details for the claimed construction. Generalizing the above constructions to a more general setting, we map $Sh_{\widehat{\mathfrak{gl}}_{m+n}} \to Sh_{\widehat{\mathfrak{gl}}_n}$. Assuming $n>2$, we want to construct algebra homomorphism $\Upsilon_{m+n,n}$:

$$\Upsilon_{m+n,n} : \bigoplus_{k_0,k_1,\ldots,k_{n-1}} Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{k_0,\ldots,k_0}_{m+1},\underbrace{k_1,\ldots,k_{n-1}}_{n-1})|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_n}(\underbrace{k_0,\ldots,k_{n-1}}_{n})|_{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3}$$

For every $f\left(X_{0,1}^{0,k_0}, X_{1,1}^{1,k_0},\ldots,X_{m,1}^{m,k_0},X_{m+1,1}^{m+1,k_1},\ldots,X_{m+n-1,1}^{m+n-1,k_{n-1}}\right)\in Sh_{\widehat{\mathfrak{gl}}_{m+n}}$, the algebra homomorphism of our *Main Theorem* is as follows:

Theorem 3.14 (Main Theorem) *The mapping:*

We have algebra homomorphism:

$$\Upsilon_{m+n,n} : \bigoplus_{k_0,k_1,\ldots,k_n-1} Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{k_0,\ldots,k_0}_{m+1},\underbrace{k_1,\ldots,k_{n-1}}_{n-1})|_{q_1,q_2,q_3} \to Sh_{\widehat{\mathfrak{gl}}_n}(\underbrace{k_0,\ldots,k_{n-1}}_{n})|_{\tilde{q}_1,\tilde{q}_2,\tilde{q}_3}$$

satisfy component-wise homomorphism via:

$$\Upsilon_{m+n,n} : f\left(X_{0,1}^{0,k_0}, X_{1,1}^{1,k_0}, \ldots, X_{m,1}^{m,k_0}, X_{m+1,1}^{m+1,k_1}, \ldots, X_{m+n-1,1}^{m+n-1,k_{n-1}}\right) \to$$
$$f\left(Z_{0,1}^{0,k_0}, \lambda_1 Z_{0,1}^{0,k_0}, \ldots, Z_{0,1}^{0,k_0}\prod_{i=1}^{m}\lambda_i, \mu_1 Z_{1,1}^{1,k_1}, \ldots, Z_{n-1,1}^{n-1,k_{n-1}}\prod_{i=1}^{n-1}\mu_i\right)\cdot G \quad (20)$$

for $1 \le i \le m, \quad \lambda_i \in \{q^{-1}d^{-1}, qd^{-1}\}$.

Proof.

We first assume
$$\lambda_1 = \lambda_2 = \ldots = \lambda_m$$

3.14.1 Case 1: $\lambda_1 = \lambda_2 =, \ldots, = \lambda_m = q^{-1}d^{-1}$

The rational factor of $G\left(z_{0,1}^{0,k_0}, \ldots, z_{n-1,1}^{n-1,k_{n-1}}\right)$ should be of the form:

$$G\left(z_{0,1}^{0,k_0}, \ldots, z_{n-1,1}^{n-1,k_{n-1}}\right) =$$
$$\frac{\prod\limits_{1\le i,j\le k_0}(z_{0,i}-q^{-1}d^{-1}z_{0,j})^m \prod\limits_{1\le i\le k_0}^{1\le j\le k_1}(q^{-m}d^{-m}z_{0,i}-\mu_1 z_{1,j})}{\prod\limits_{1\le i\le k_0}^{1\le j\le k_1}(z_{0,i}-z_{1,j})\prod\limits_{1\le i\le k_1}^{1\le j\le k_2}(z_{1,i}-z_{2,j})\cdots\prod\limits_{1\le i\le k_{n-1}}^{1\le j\le k_0}(z_{n-1,i}-z_{0,j})}\cdot$$
$$\frac{\prod\limits_{1\le i\le k_1}^{1\le j\le k_2}(z_{1,i}-\mu_2 z_{2,j})\cdots\prod\limits_{1\le i\le k_{n-1}}^{1\le j\le k_0}(z_{n-1,i}\prod\limits_{1\le r\le n-1}\mu_r - z_{0,j})}{\prod\limits_{1\le i<j\le k_0}(z_{0,i}-q^2 z_{0,j})^m(z_{0,i}-q^{-2}z_{0,j})^m}\cdot C_{k_0}^{k_{n-1}} \quad (21)$$

For $f \in Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{k_0,\ldots,k_0,}_{m+1}\underbrace{k_1,\ldots,k_{n-1}}_{n-1})$ and $g \in Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{l_0,\ldots,l_0,}_{m+1}\underbrace{l_1,\ldots,l_{n-1}}_{n-1})$, that is

$$f\left(Z_{0,1}^{0,k_0}, \frac{1}{qd}Z_{0,1}^{0,k_0}, \ldots, \frac{1}{q^m d^m}Z_{0,1}^{0,k_0}, \mu_1 Z_{1,1}^{1,k_1}, \ldots, Z_{n-1,1}^{n-1,k_{n-1}}\prod_{r=1}^{n-1}\mu_r\right)$$

$$g\left(Z_{0,k_0+1}^{0,k_0+l_0}, \frac{1}{qd}Z_{0,k_0+1}^{0,k_0+l_0}, \ldots, \frac{1}{q^m d^m}Z_{0,k_0+1}^{0,k_0+l_0}, \mu_1 Z_{1,k_1+1}^{1,k_1+l_1}, \ldots, Z_{n-1,k_{n-1}+1}^{n-1,k_{n-1}+l_{n-1}}\prod_{r=1}^{n-1}\mu_r\right)$$

The specialization of $\omega^{(m+n)}\left(\frac{z_{0,1}^{0,k_0},\ldots,z_{n-1,1}^{n-1,k_{n-1}}\prod_{r=1}^{n-1}\mu_r}{z_{0,k_0+1}^{0,k_0+l_0},\ldots,z_{n-1,k_{n-1}+1}^{n-1,k_{n-1}+l_{n-1}}\prod_{r=1}^{n-1}\mu_r}\right)$ is:

$$
\begin{aligned}
&\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\left(\frac{z_{0,i}-q^{-2}z_{0,j}}{z_{0,i}-z_{0,j}}\right)^{m+1}\left(\frac{d^{-1}z_{0,i}-d^{-1}z_{0,j}}{z_{0,i}-q^{-1}d^{-1}z_{0,j}}\right)^{m}\left(\frac{z_{0,i}-q^{2}z_{0,j}}{z_{0,i}-qdz_{0,j}}\right)^{m}\cdot\\
&\prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0}\frac{\mu_1 z_{1,i}-q^{-(m-1)}d^{-(m+1)}z_{0,j}}{\mu_1 z_{1,i}-q^{-m}d^{-m}z_{0,j}}\cdot\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\frac{d^{-1}q^{-m}d^{-m}z_{0,i}-q\mu_1 z_{1,j}}{q^{-m}d^{-m}z_{0,i}-\mu_1 z_{1,j}}\cdot\\
&\prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0}\frac{z_{n-1,i}d^{-1}\prod_{r=1}^{n-1}\mu_r-qz_{0,j}}{z_{n-1,i}\prod_{r=1}^{n-1}\mu_r-z_{0,j}}\cdot\prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}}\frac{z_{0,i}-qd^{-1}z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}{z_{0,i}-z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}\cdot\\
&\prod_{a=1}^{n-1}\prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a}\frac{z_{a,i}-q^{-2}z_{a,j}}{z_{a,i}-z_{a,j}}\cdot\prod_{a=1}^{n-2}\prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}}\frac{d^{-1}z_{a,i}-q\mu_{a+1}z_{a+1,j}}{z_{a,i}-\mu_{a+1}z_{a+1,j}}\cdot\\
&\prod_{a=1}^{n-2}\prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a}\frac{\mu_{a+1}z_{a+1,i}-qd^{-1}z_{a,j}}{\mu_{a+1}z_{a+1,i}-z_{a,j}}
\end{aligned}\tag{22}
$$

Pulling out the constants of (21), the ratio of G is:

$$
\begin{aligned}
&\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\frac{\left(z_{0,i}-q^{-1}d^{-1}z_{0,j}\right)^m\left(z_{0,j}-q^{-1}d^{-1}z_{0,i}\right)^m}{\left(z_{0,i}-q^{2}z_{0,j}\right)^m\left(z_{0,i}-q^{-2}z_{0,j}\right)^m}\cdot\\
&\frac{\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\left(q^{-m}d^{-m}z_{0,i}-\mu_1 z_{1,j}\right)\prod_{k_0<i\le k_0+l_0}^{1\le j\le k_1}\left(q^{-m}d^{-m}z_{0,i}-\mu_1 z_{1,j}\right)}{\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\left(z_{0,i}-z_{1,j}\right)\prod_{k_0<i\le k_0+l_0}^{1\le j\le k_1}\left(z_{0,i}-z_{1,j}\right)}\cdot\\
&\frac{\prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0}\left(z_{n-1,i}\prod_{r=1}^{n-1}\mu_r-z_{0,j}\right)\prod_{k_{n-1}<i\le k_{n-1}+l_{n-1}}^{1\le j\le k_0}\left(z_{n-1,i}\prod_{r=1}^{n-1}\mu_r-z_{0,j}\right)}{\prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0}\left(z_{n-1,i}-z_{0,j}\right)\prod_{k_{n-1}<i\le k_{n-1}+l_{n-1}}^{1\le j\le k_0}\left(z_{n-1,i}-z_{0,j}\right)}\cdot\\
&\frac{\prod_{a=1}^{n-2}\prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}}\left(z_{a,i}-\mu_{a+1}z_{a+1,j}\right)\cdot\prod_{a=1}^{n-2}\prod_{k_a<i\le k_a+l_a}^{1\le j\le k_{a+1}}\left(z_{a,i}-\mu_{a+1}z_{a+1,j}\right)}{\prod_{a=1}^{n-2}\prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}}\left(z_{a,i}-z_{a+1,j}\right)\cdot\prod_{a=1}^{n-2}\prod_{k_a<i\le k_a+l_a}^{1\le j\le k_{a+1}}\left(z_{a,i}-z_{a+1,j}\right)}
\end{aligned}\tag{23}
$$

Calculating the product of (22) and (23) involves massive calculations. But we can determine $\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q},\tilde{d}$ and G by analyzing $\rho(z_{u,v},z_{s,t})$ which is the product of terms containing $(z_{u,v},z_{s,t})$, $0\le u,s\le n-1$, $1\le v,t\le \max(k_r+l_r)$, $1\le r\le n-1$.

1. $\rho(z_{0,i}, z_{0,j})$ in the product is:

$$\rho(z_{0,i}, z_{0,j}) = \prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0} \left(\frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}}\right)^{m+1} \cdot \left(\frac{d^{-1}z_{0,i} - d^{-1}z_{0,j}}{z_{0,i} - q^{-1}d^{-1}z_{0,j}}\right)^{m} \cdot$$
$$\left(\frac{z_{0,i} - q^{2}z_{0,j}}{z_{0,i} - qdz_{0,j}}\right)^{m} \cdot \frac{(z_{0,i} - q^{-1}d^{-1}z_{0,j})^m (z_{0,j} - q^{-1}d^{-1}z_{0,i})^m}{(z_{0,i} - q^{2}z_{0,j})^m (z_{0,i} - q^{-2}z_{0,j})^m}$$
$$\rho(z_{0,i}, z_{0,j}) = (-q^{-1}d^{-2})^{k_0 l_0} \prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0} \frac{z_{0,i} - q^{-2}z_{0,j}}{z_{0,i} - z_{0,j}}$$

We know $\frac{z_{0,i}-q^{-2}z_{0,j}}{z_{0,i}-z_{0,j}}$ is the shuffle factors of $\omega_{i,i}^{(n)}\left(\frac{z_{0,i}}{z_{0,j}}\right)$, when $\tilde{q} = q$.

2. $\rho(z_{0,i}, z_{1,j})$ with $1 \le i \le k_0, k_1 < j \le k_1 + l_1$in the product is:

$$\rho(z_{0,i}, z_{1,j}) = \prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1} \frac{z_{0,i} - q^{m+1}d^{m+1}\mu_1 z_{1,j}}{z_{0,i} - q^{m}d^{m}\mu_1 z_{1,j}} \cdot \frac{z_{0,i} - q^{m}d^{m}\mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \frac{1}{q^m d^{m+1}}$$
$$\rho(z_{0,i}, z_{1,j}) = \prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1} \left(\frac{z_{0,i} - q^{m+1}d^{m+1}\mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \frac{1}{q^m d^{m+1}}\right)$$

In order to identify $\rho(z_{0,i}, z_{1,j})$ with $const \cdot \prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1} \omega_{i,i+1}^{(n)}\left(\frac{z_{0,i}}{z_{1,j}}\right)$, we must have $\tilde{q}\tilde{d} = q^{m+1}d^{m+1}\mu_1$.

3. $\rho(z_{1,i}, z_{0,j})$ with $1 \le i \le k_1, k_0 < j \le k_0 + l_0$ in the product is:

$$\rho(z_{1,i}, z_{0,j}) = \prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \frac{z_{1,i} - q^{-(m-1)}d^{-(m+1)}\mu_1^{-1}z_{0,j}}{z_{1,i} - q^{-m}d^{-m}\mu_1^{-1}z_{0,j}} \cdot$$
$$\frac{z_{1,i} - q^{-m}d^{-m}\mu_1^{-1}z_{0,j}}{z_{1,i} - z_{0,j}} \cdot \mu_1$$
$$\rho(z_{1,i}, z_{0,j}) = \prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \left(\frac{z_{1,i} - q^{-(m-1)}d^{-(m+1)}\mu_1^{-1}z_{0,j}}{z_{1,i} - z_{0,j}} \cdot \mu_1\right)$$

In order to identify $\rho(z_{1,i}, z_{0,j})$ with $const \cdot \prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \omega_{i,i-1}^{(n)}\left(\frac{z_{1,i}}{z_{0,j}}\right)$, we must have $\tilde{q}\tilde{d}^{-1} = q^{-(m-1)}d^{-(m+1)}\mu_1^{-1}$.
Combining $\tilde{q}\tilde{d} = q^{m+1}d^{m+1}\mu_1$ and $\tilde{q}\tilde{d}^{-1} = q^{-(m-1)}d^{-(m+1)}\mu_1^{-1}$, we have $\tilde{d} = q^m d^{m+1}\mu_1$, as $q = \tilde{q}$.

4. $\rho(z_{0,i}, z_{n-1,j})$ with $1 \le i \le k_0$, $k_{n-1} < j \le k_{n-1} + l_{n-1}$ in the product is:

$$\rho(z_{0,i}, z_{n-1,j}) = \prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \frac{z_{0,i} - qd^{-1}z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}\prod_{r=1}^{n-1}\mu_r} \cdot \frac{z_{0,i} - z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}}$$

$$\rho(z_{0,i}, z_{n-1,j}) = \prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \frac{z_{0,i} - qd^{-1}z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}}$$

In order to identify $\rho(z_{0,i}, z_{n-1,j})$ with $\prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \omega^{(n)}_{i,i-1}\left(\frac{z_{0,i}}{z_{n-1,j}}\right)$, we must have $\tilde{q}\tilde{d}^{-1} = qd^{-1}\prod_{r=1}^{n-1}\mu_r$, which means $\tilde{q} = q$ and $\tilde{d} = d\prod_{r=1}^{n-1}\mu_r^{-1}$.

5. $\rho(z_{n-1,i}, z_{0,j})$ with $1 \le i \le k_{n-1}$, $k_0 < j \le k_0 + l_0$ in the product is:

$$\rho(z_{n-1,i}, z_{0,j}) = \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \frac{z_{n-1,i} - qdz_{0,j}\prod_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}\prod_{r=1}^{n-1}\mu_r^{-1}} \cdot \frac{z_{n-1,i} - z_{0,j}\prod_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot d^{-1}\prod_{r=1}^{n-1}\mu_r$$

$$\rho(z_{n-1,i}, z_{0,j}) = \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \left(\frac{z_{n-1,i} - qdz_{0,j}\prod_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot d^{-1}\prod_{r=1}^{n-1}\mu_r \right)$$

In order to identify $\rho(z_{n-1,i}, z_{0,j})$ with $const \cdot \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \omega^{(n)}_{i,i+1}\left(\frac{z_{n-1,i}}{z_{0,j}}\right)$, we must have $\tilde{q}\tilde{d} = qd\prod_{r=1}^{n-1}\mu_r^{-1}$.

6. $\rho(z_{a,i}, z_{a,j})$ with $1 \le i \le k_a < j \le k_a + l_a$ in the product is:

$$\rho(z_{a,i}, z_{a,j}) = \prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a} \frac{z_{a,i} - q^{-2}z_{a,j}}{z_{a,i} - z_{a,j}} = \prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a} \omega^{(n)}_{i,i}\left(\frac{z_{a,i}}{z_{a,j}}\right)$$

given $\tilde{q} = q$.

7. $\rho(z_{a,i}, z_{a+1,j})$ with $1 \le i \le k_a$, $k_{a+1} < j \le k_{a+1} + l_{a+1}, 1 \le a \le n-2$ in the product is:

$$\rho(z_{a,i}, z_{a+1,j}) = \prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \frac{z_{a,i} - qd\mu_{a+1}z_{a+1,j}}{z_{a,i} - \mu_{a+1}z_{a+1,j}} \cdot \frac{1}{d} \cdot \frac{z_{a,i} - \mu_{a+1}z_{a+1,j}}{z_{a,i} - z_{a+1,j}}$$

$$\rho(z_{a,i}, z_{a+1,j}) = \prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \left(\frac{z_{a,i} - qd\mu_{a+1}z_{a+1,j}}{z_{a,i} - z_{a+1,j}} \cdot \frac{1}{d}\right)$$

In order to identify $\rho(z_{a,i}, z_{a+1,j})$ with $\prod\limits_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \omega^{(n)}_{i,i+1}\left(\frac{z_{a,i}}{z_{a+1,j}}\right)$, we must have $\tilde{q}\tilde{d} = qd\mu_{a+1}$ which means $\tilde{q} = q$ and $\tilde{d} = d\mu_{a+1}$.

8. $\rho(z_{a+1,i}, z_{a,j})$ with $1 \le i \le k_{a+1}$, $k_a < j \le k_a + l_a, 1 \le a \le n-2$ in the product is:

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \frac{z_{a+1,i} - qd^{-1}\mu_{a+1}^{-1}z_{a,j}}{z_{a+1,i} - \mu_{a+1}^{-1}z_{a,j}} \cdot \frac{z_{a+1,i} - \mu_{a+1}^{-1}z_{a,j}}{z_{a+1,i} - z_{a,j}} \cdot \mu_{a+1}$$

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \left(\frac{z_{a+1,i} - qd^{-1}\mu_{a+1}^{-1}z_{a,j}}{z_{a+1,i} - z_{a,j}} \cdot \mu_{a+1}\right)$$

In order to identify $\rho(z_{a+1,i}, z_{a,j})$ with $\prod\limits_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \omega^{(n)}_{i,i-1}\left(\frac{z_{a+1,i}}{z_{a,j}}\right)$, we must have $\tilde{q}\tilde{d}^{-1} = qd^{-1}\mu_{a+1}^{-1}$ which is guaranteed by $\rho(z_{a,i}, z_{a+1,j})$.
To summarize, we have obtained that:

$$\tilde{d} = q^m d^{m+1}\mu_1, \quad \tilde{d} = d\mu_2, \ldots, \tilde{d} = d\mu_{n-1}, \quad \tilde{d} = d\prod_{r=1}^{n-1}\mu_r^{-1}$$

Taking the product, we have: $\tilde{d}^n = q^m d^{m+n}$ and $\tilde{d} = q^{\frac{m}{n}} d^{\frac{m}{n}+1} = dq_3^{-\frac{m}{n}}$.
Since $q_1 = dq^{-1}, q_3 = d^{-1}q^{-1}$, we have $d = q_1^{\frac{1}{2}}q_3^{-\frac{1}{2}}, q = q_1^{-\frac{1}{2}}q_3^{-\frac{1}{2}}$. Therefore,

$$\begin{aligned}
&\tilde{q}_1 = q_1 q_3^{-\frac{m}{n}}, \quad \tilde{q}_2 = q_2, \quad \tilde{q}_3 = q_3 q_3^{\frac{m}{n}} \\
&\mu_2 = \mu_3 =, \ldots, = \mu_{n-1} = \tilde{d}d^{-1} = q^{\frac{m}{n}} d^{\frac{m}{n}} = q_3^{-\frac{m}{n}} \\
&\mu_1 = \tilde{d}q^{-m}d^{-(m+1)} = (qd)^{\frac{m}{n}-m} = q_3^{m-\frac{m}{n}}
\end{aligned}$$

Now, let's determine the constants C_k. We know that

$$\frac{C_{k+l}}{C_k C_l} \cdot \left(-\frac{1}{qd^2}\right)^{k_0 l_0} \mu_1^{k_1 l_0} \cdot \mu_2^{k_2 l_1} \cdots \mu_{n-1}^{k_{n-1} l_{n-2}} \cdot \left(\frac{\tilde{d}}{q^m d^{m+1}}\right)^{k_0 l_1} \left(\frac{\tilde{d}}{d}\right)^{\sum\limits_{a=1}^{n-2} k_a l_{a+1}} = 1$$

$$\frac{\tilde{d}}{q^m d^{m+1}} = \mu_1, \text{ and } \mu_2 =, \ldots, = \mu_{n-1} = \frac{\tilde{d}}{d}$$

Therefore,

$$\frac{C_{k+l}}{C_k C_l}\prod_{a=1}^{n-1}\mu_a^{k_a l_{a-1}+k_{a-1}l_a}\cdot\left(-\frac{1}{qd^2}\right)^{k_0 l_0}=1$$

In particular, we can choose $C_k=\prod_{a=1}^{n-1}\mu_a^{-k_a l_{a-1}}\cdot(-qd^2)^{\frac{k_0^2}{2}}$.
It remains to verify wheel conditions:

1. The $Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}$ wheel in color $(r,r,r+1)$ $1\le r\le n-2$ implies vanishing as $q_3^{\frac{m}{n}}\cdot\frac{z_{r,a}}{z_{r+1,b}}=qd, q_3^{-\frac{m}{n}}\cdot\frac{z_{r+1,b}}{z_{r,c}}=qd^{-1}$, which means $\frac{z_{r,a}}{z_{r+1,b}}=\tilde{q}\tilde{d}, \frac{z_{r+1,b}}{z_{r,c}}=\tilde{q}\tilde{d}^{-1}$.

2. The $Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}$ wheel in color $(0,0,1)$ and $(n-1,n-1,0)$are verified as above by using $\tilde{d}d^{-1}=\prod_{1\le r\le n-1}\mu_r^{-1}=q_3^{-\frac{m}{n}}, \mu_1 q_3^{-m}=q_3^{-\frac{m}{n}}$.

3. The $Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}$ wheel in color $(r,r+1,r+1)$implies vanishing as $q_3^{-\frac{m}{n}}\cdot\frac{z_{r+1,a}}{z_{r,b}}=qd^{-1}, q_3^{\frac{m}{n}}\cdot\frac{z_{r,b}}{z_{r+1,c}}=qd$, which means $\frac{z_{r,b}}{z_{r+1,c}}=\tilde{q}\tilde{d}, \frac{z_{r+1,a}}{z_{r,b}}=\tilde{q}\tilde{d}^{-1}$.

4. The $Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}$ wheel in color $(n-1,0,0)$ and $(0,1,0)$are verified in the same way.

Therefore, we have constructed a homomorphism $\Upsilon_{m+n,n}$

$$\Upsilon_{m+n,n}:\bigoplus_{k_0,k_1,\dots,k_{n-1}} Sh_{\widehat{\widehat{\mathfrak{gl}}}_{m+n}}(\underbrace{k_0,\dots,k_0}_{m+1},\underbrace{k_1,\dots,k_{n-1}}_{n-1})|_{q_1,q_2,q_3}\to$$
$$Sh_{\widehat{\widehat{\mathfrak{gl}}}_n}(\underbrace{k_0,\dots,k_{n-1}}_{n})|_{\tilde{q}_1=q_1q_3^{-\frac{m}{n}},\tilde{q}_2=q_2,\tilde{q}_3=q_3q_3^{\frac{m}{n}}} \qquad \text{given by}$$
$$\Upsilon_{m+n,n}:f\left(X_{0,1}^{0,k_0},X_{1,1}^{1,k_0},\dots,X_{m,1}^{m,k_0},X_{m+1,1}^{m+1,k_1},\dots,X_{m+n-1,1}^{m+n-1,k_{n-1}}\right)\to$$
$$f\left(Z_{0,1}^{0,k_0},q_3Z_{0,1}^{0,k_0},\dots,q_3^mZ_{0,1}^{0,k_0},q_3^{m-\frac{m}{n}}Z_{1,1}^{1,k_1},\dots,q_3^{\frac{m}{n}}Z_{n-1,1}^{n-1,k_{n-1}}\right)\cdot G \quad \text{with}$$

$$G=\frac{\prod_{1\le i,j\le k_0}(z_{0,i}-q^{-1}d^{-1}z_{0,j})^m\cdot\prod_{a=0}^{n-1}\prod_{1\le i\le k_a}^{1\le j\le k_{a+1}}(z_{a,i}-q_3^{-\frac{m}{n}}z_{a+1,j})}{\prod_{1\le i,j\le k_0}((z_{0,i}-q^2z_{0,j})^m\cdot(z_{0,i}-q^{-2}z_{0,j})^m)}\cdot$$
$$\frac{\prod_{a=1}^{n-1}\mu_a^{-k_a l_{a-1}}\cdot(-qd^2)^{\frac{k_0^2}{2}}}{\prod_{a=0}^{n-1}\prod_{1\le i\le k_a}^{1\le j\le k_{a+1}}(z_{a,i}-z_{a+1,j})}$$

3.14.2 Case 2: $\lambda_1 = \lambda_2 =, \dots, = \lambda_m = qd^{-1}$

The rational factor of $G\left(z_{0,1}^{0,k_0}, \dots, z_{n-1,1}^{n-1,k_{n-1}}\right)$ should be of the form:

$$G\left(z_{0,1}^{0,k_0}, \dots, z_{n-1,1}^{n-1,k_{n-1}}\right) =$$

$$\frac{\prod\limits_{1\le i,j\le k_0} (z_{0,i} - qd^{-1} z_{0,j})^m \cdot \prod\limits_{1\le i\le k_0}^{1\le j\le k_1} (q^m d^{-m} z_{0,i} - \mu_1 z_{1,j})}{\prod\limits_{1\le i\le k_0}^{1\le j\le k_1} (z_{0,i} - z_{1,j}) \cdot \prod\limits_{1\le i\le k_1}^{1\le j\le k_2} (z_{1,i} - z_{2,j}) \cdots \prod\limits_{1\le i\le k_{n-1}}^{1\le j\le k_0} (z_{n-1,i} - z_{0,j})} \cdot$$

$$\frac{\prod\limits_{1\le i\le k_1}^{1\le j\le k_2} (z_{1,i} - \mu_2 z_{2,j}) \cdots \prod\limits_{1\le i\le k_{n-1}}^{1\le j\le k_0} \left(z_{n-1,i} \prod\limits_{1\le r\le n-1} \mu_r - z_{0,j}\right)}{\prod\limits_{1\le i<j\le k_0} (z_{0,i} - q^2 z_{0,j})^m (z_{0,i} - q^{-2} z_{0,j})^m} \cdot C_k$$

For $f \in Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{k_0, \dots, k_0}_{m+1}, \underbrace{k_1, \dots, k_{n-1}}_{n-1})$ and $g \in Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{l_0, \dots, l_0}_{m+1}, \underbrace{l_1, \dots, l_{n-1}}_{n-1})$,

That is

$$f\left(Z_{0,1}^{0,k_0}, \frac{q}{d} Z_{0,1}^{0,k_0}, \dots, \frac{q^m}{d^m} Z_{0,1}^{0,k_0}, \mu_1 Z_{1,1}^{1,k_1}, \dots, Z_{n-1,1}^{n-1,k_{n-1}} \prod_{r=1}^{n-1} \mu_r\right)$$

$$g\left(Z_{0,k_0+1}^{0,k_0+l_0}, \frac{q}{d} Z_{0,k_0+1}^{0,k_0+l_0}, \dots, \frac{q^m}{d^m} Z_{0,k_0+1}^{0,k_0+l_0}, \mu_1 Z_{1,k_1+1}^{1,k_1+l_1}, \dots, Z_{n-1,k_{n-1}+1}^{n-1,k_{n-1}+l_{n-1}} \prod_{r=1}^{n-1} \mu_r\right)$$

The ratio of factor of G ignoring the constants, is:

$$\begin{aligned}
&\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0} \left(\frac{z_{0,i} - qd^{-1} z_{0,j}}{z_{0,i} - q^2 z_{0,j}} \cdot \frac{z_{0,j} - qd^{-1} z_{0,i}}{z_{0,i} - q^{-2} z_{0,j}}\right)^m \cdot \\
&\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1} \frac{q^m d^{-m} z_{0,i} - \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \prod_{k_0<i\le k_0+l_0}^{1\le j\le k_1} \frac{q^m d^{-m} z_{0,i} - \mu_1 z_{1,j}}{z_{0,i} - z_{1,j}} \cdot \\
&\prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \frac{z_{n-1,i} \prod\limits_{r=1}^{n-1} \mu_r - z_{0,j}}{z_{n-1,i} - z_{0,j}} \cdot \prod_{k_{n-1}<i\le k_{n-1}+l_{n-1}}^{1\le j\le k_0} \frac{z_{n-1,i} \prod\limits_{r=1}^{n-1} \mu_r - z_{0,j}}{z_{n-1,i} - z_{0,j}} \cdot \\
&\prod_{a=1}^{n-2} \prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \frac{z_{a,i} - \mu_{a+1} z_{a+1,j}}{z_{a,i} - z_{a+1,j}} \cdot \prod_{a=1}^{n-2} \prod_{k_a<i\le k_a+l_a}^{1\le j\le k_{a+1}} \frac{z_{a,i} - \mu_{a+1} z_{a+1,j}}{z_{a,i} - z_{a+1,j}}
\end{aligned} \tag{24}$$

The specialization of $\omega^{(n)}\left(\frac{z_{0,1}^{0,k_0}, \dots, z_{n-1,1}^{n-1,k_{n-1}} \prod\limits_{r=1}^{n-1} \mu_r}{z_{0,k_0+1}^{0,k_0+l_0}, \dots, z_{n-1,k_{n-1}+1}^{n-1,k_{n-1}+l_{n-1}} \prod\limits_{r=1}^{n-1} \mu_r}\right)$ is:

$$\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\left(\frac{z_{0,i}-q^{-2}z_{0,j}}{z_{0,i}-z_{0,j}}\right)^{m+1}\left(\frac{d^{-1}z_{0,i}-q^2d^{-1}z_{0,j}}{z_{0,i}-qd^{-1}z_{0,j}}\right)^{m}\left(\frac{qd^{-1}z_{0,i}-qd^{-1}z_{0,j}}{qd^{-1}z_{0,i}-z_{0,j}}\right)^{m}\cdot$$

$$\prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0}\frac{\mu_1 z_{1,i}-q^{m+1}d^{-(m+1)}z_{0,j}}{\mu_1 z_{1,i}-q^m d^{-m}z_{0,j}}\cdot\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\frac{q^m d^{-(m+1)}z_{0,i}-q\mu_1 z_{1,j}}{q^m d^{-m}z_{0,i}-\mu_1 z_{1,j}}\cdot$$

$$\prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0}\frac{d^{-1}z_{n-1,i}\prod_{r=1}^{n-1}\mu_r-qz_{0,j}}{z_{n-1,i}\prod_{r=1}^{n-1}\mu_r-z_{0,j}}\cdot\prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}}\frac{z_{0,i}-qd^{-1}z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}{z_{0,i}-z_{n-1,j}\prod_{r=1}^{n-1}\mu_r}\cdot$$

$$\prod_{a=1}^{n-1}\prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a}\frac{z_{a,i}-q^{-2}z_{a,j}}{z_{a,i}-z_{a,j}}\cdot\prod_{a=1}^{n-2}\prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}}\frac{d^{-1}z_{a,i}-q\mu_{a+1}z_{a+1,j}}{z_{a,i}-\mu_{a+1}z_{a+1,j}}\cdot$$

$$\prod_{a=1}^{n-2}\prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a}\frac{\mu_{a+1}z_{a+1,i}-qd^{-1}z_{a,j}}{\mu_{a+1}z_{a+1,i}-z_{a,j}} \tag{25}$$

We can determine $\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q},\tilde{d}$ and G using the same method as above cases by analysing the product of terms in (25) and (24):

1. $\rho(z_{0,i},z_{0,j})$ in the product is:

$$\rho(z_{0,i},z_{0,j})=\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\left(\frac{z_{0,i}-q^{-2}z_{0,j}}{z_{0,i}-z_{0,j}}\right)^{m+1}\cdot\left(\frac{z_{0,i}-q^2z_{0,j}}{z_{0,i}-qd^{-1}z_{0,j}}\right)^{m}\cdot$$

$$\left(\frac{z_{0,i}-z_{0,j}}{z_{0,i}-q^{-1}dz_{0,j}}\right)^{m}\cdot\frac{\left(z_{0,i}-qd^{-1}z_{0,j}\right)^m\left(z_{0,i}-q^{-1}dz_{0,j}\right)^m}{\left(z_{0,i}-q^2z_{0,j}\right)^m\left(z_{0,i}-q^{-2}z_{0,j}\right)^m}\cdot(-qd^{-2})^{k_0l_0}$$

$$\rho(z_{0,i},z_{0,j})=(-qd^{-2})^{k_0l_0}\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\frac{z_{0,i}-q^{-2}z_{0,j}}{z_{0,i}-z_{0,j}}$$

If $\tilde{q}=q$ then

$$\rho(z_{0,i},z_{0,j})=(-qd^{-2})^{k_0l_0}\prod_{1\le i\le k_0}^{k_0<j\le k_0+l_0}\omega_{i,i}^{(n)}\left(\frac{z_{0,i}}{z_{0,j}}\right)$$

2. $\rho(z_{0,i},z_{1,j})$ with $1\le i\le k_0, k_1\le j\le k_1+l_1$in the product is:

$$\rho(z_{0,i},z_{1,j})=\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\frac{z_{0,i}-q^{-(m-1)}d^{m+1}\mu_1z_{1,j}}{z_{0,i}-q^{-m}d^m\mu_1z_{1,j}}\cdot\frac{z_{0,i}-q^md^m\mu_1z_{1,j}}{z_{0,i}-z_{1,j}}\cdot\frac{q^m}{d^{m+1}}$$

$$\rho(z_{0,i},z_{1,j})=\prod_{1\le i\le k_0}^{k_1<j\le k_1+l_1}\left(\frac{z_{0,i}-q^{-(m-1)}d^{m+1}\mu_1z_{1,j}}{z_{0,i}-z_{1,j}}\cdot\frac{q^m}{d^{m+1}}\right)$$

In order to identify $\rho(z_{0,i}, z_{1,j})$ with $const \cdot \prod\limits_{1\le i\le k_0}^{k_1<j\le k_1+l_1} \omega^{(n)}_{i,i+1}\left(\frac{z_{0,i}}{z_{1,j}}\right)$, we must have $\tilde{q}\tilde{d} = q^{-(m-1)}d^{m+1}\mu_1$. That is $\tilde{d} = q^{-m}d^{m+1}\mu_1$

3. $\rho(z_{1,i}, z_{0,j})$ with $1 \le i \le k_1, k_0 < j \le k_0 + l_0$ in the product is:

$$\rho(z_{1,i}, z_{0,j}) = \prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \frac{z_{1,i} - q^{m+1}d^{-(m+1)}\mu_1^{-1}z_{0,j}}{z_{1,i} - q^m d^{-m}\mu_1^{-1}z_{0,j}} \cdot \frac{z_{1,i} - q^m d^{-m}\mu_1^{-1}z_{0,j}}{z_{1,i} - z_{0,j}} \cdot \mu_1$$

$$\rho(z_{1,i}, z_{0,j}) = \prod_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \left(\frac{z_{1,i} - q^{m+1}d^{-(m+1)}\mu_1^{-1}z_{0,j}}{z_{1,i} - z_{0,j}} \cdot \mu_1\right)$$

In order to identify $\rho(z_{1,i}, z_{0,j})$ with $const \cdot \prod\limits_{1\le i\le k_1}^{k_0<j\le k_0+l_0} \omega^{(n)}_{i,i-1}\left(\frac{z_{1,i}}{z_{0,j}}\right)$, we must have
$\tilde{q}\tilde{d}^{-1} = q^{m+1}d^{-(m+1)}\mu_1^{-1}$.

4. $\rho(z_{0,i}, z_{n-1,j})$ with $1 \le i \le k_0$, $k_{n-1} < j \le k_{n-1} + l_{n-1}$ in the product is:

$$\rho(z_{0,i}, z_{n-1,j}) = \prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \frac{z_{0,i} - qd^{-1}z_{n-1,j}\prod\limits_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}\prod\limits_{r=1}^{n-1}\mu_r} \cdot \frac{z_{0,i} - z_{n-1,j}\prod\limits_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}}$$

$$\rho(z_{0,i}, z_{n-1,j}) = \prod_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \frac{z_{0,i} - qd^{-1}z_{n-1,j}\prod\limits_{r=1}^{n-1}\mu_r}{z_{0,i} - z_{n-1,j}}$$

In order to identify $\rho(z_{0,i}, z_{n-1,j})$ with $\prod\limits_{1\le i\le k_0}^{k_{n-1}<j\le k_{n-1}+l_{n-1}} \omega^{(n)}_{i,i-1}\left(\frac{z_{0,i}}{z_{n-1,j}}\right)$, we must have $\tilde{q}\tilde{d}^{-1} = qd^{-1}\prod\limits_{r=1}^{n-1}\mu_r$, which means $\tilde{d} = d\prod\limits_{r=1}^{n-1}\mu_r^{-1}$.

5. $\rho(z_{n-1,i}, z_{0,j})$ with $1 \le i \le k_{n-1}$, $k_0 < j \le k_0 + l_0$ in the product is:

$$\rho(z_{n-1,i}, z_{0,j}) = \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \frac{z_{n-1,i} - qdz_{0,j}\prod\limits_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}\prod\limits_{r=1}^{n-1}\mu_r^{-1}} \cdot \frac{z_{n-1,i} - z_{0,j}\prod\limits_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot d^{-1}\prod_{r=1}^{n-1}\mu_r$$

$$\rho(z_{n-1,i}, z_{0,j}) = \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \left(\frac{z_{n-1,i} - qdz_{0,j}\prod\limits_{r=1}^{n-1}\mu_r^{-1}}{z_{n-1,i} - z_{0,j}} \cdot d^{-1}\prod_{r=1}^{n-1}\mu_r\right)$$

In order to identify $\rho(z_{n-1,i}, z_{0,j})$ with $const \cdot \prod_{1\le i\le k_{n-1}}^{k_0<j\le k_0+l_0} \omega^{(n)}_{i,i+1}\left(\frac{z_{n-1,i}}{z_{0,j}}\right)$, we must have $\tilde{q}\tilde{d} = qd\prod_{r=1}^{n-1}\mu_r^{-1}$, which is guaranteed by $\rho(z_{0,i}, z_{n-1,j})$.

6. $\rho(z_{a,i}, z_{a,j})$ with $1 \le i \le k_a < j \le k_a + l_a$ in the product is:

$$\rho(z_{a,i}, z_{a,j}) = \prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a} \frac{z_{a,i} - q^{-2}z_{a,j}}{z_{a,i} - z_{a,j}} = \prod_{1\le i\le k_a}^{k_a<j\le k_a+l_a} \omega^{(n)}_{i,i}\left(\frac{z_{a,i}}{z_{a,j}}\right)$$

   We have $\tilde{q} = q$.

7. $\rho(z_{a,i}, z_{a+1,j})$ with $1 \le i \le k_a,\ k_{a+1} < j \le k_{a+1} + l_{a+1} \quad \forall 1 \le a \le n-2$ in the product is:

$$\rho(z_{a,i}, z_{a+1,j}) = \prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \frac{z_{a,i} - qd\mu_{a+1}z_{a+1,j}}{z_{a,i} - \mu_{a+1}z_{a+1,j}} \cdot \frac{1}{d} \cdot \frac{z_{a,i} - \mu_{a+1}z_{a+1,j}}{z_{a,i} - z_{a+1,j}}$$

$$\rho(z_{a,i}, z_{a+1,j}) = \prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \left(\frac{z_{a,i} - qd\mu_{a+1}z_{a+1,j}}{z_{a,i} - z_{a+1,j}} \cdot \frac{1}{d}\right)$$

   In order to identify $\rho(z_{a,i}, z_{a+1,j})$ with $\prod_{1\le i\le k_a}^{k_{a+1}<j\le k_{a+1}+l_{a+1}} \omega^{3}_{i,i+1}\left(\frac{z_{a,i}}{z_{a+1,j}}\right)$, we must have $\tilde{q}\tilde{d} = qd\mu_{a+1}$ which means $\tilde{d} = d\mu_{a+1}$.

8. $\rho(z_{a+1,i}, z_{a,j})$ with $1 \le i \le k_{a+1},\ k_a < j \le k_a + l_a \quad \forall 1 \le a \le n-2$ in the product is:

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \frac{z_{a+1,i} - qd^{-1}\mu^{-1}_{a+1}z_{a,j}}{z_{a+1,i} - \mu^{-1}_{a+1}z_{a,j}} \cdot \frac{z_{a+1,i} - \mu^{-1}_{a+1}z_{a,j}}{z_{a+1,i} - z_{a,j}} \cdot \mu_{a+1}$$

$$\rho(z_{a+1,i}, z_{a,j}) = \prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \left(\frac{z_{a+1,i} - qd^{-1}\mu^{-1}_{a+1}z_{a,j}}{z_{a+1,i} - z_{a,j}} \cdot \mu_{a+1}\right)$$

   In order to identify $\rho(z_{a+1,i}, z_{a,j})$ with $\prod_{1\le i\le k_{a+1}}^{k_a<j\le k_a+l_a} \omega^{(n)}_{i,i-1}\left(\frac{z_{a+1,i}}{z_{a,j}}\right)$, we must have $\tilde{q}\tilde{d}^{-1} = qd^{-1}\mu^{-1}_{a+1}$ which is guaranteed by $\rho(z_{a,i}, z_{a+1,j})$.

It remains to verify wheel conditions:

1. The $Sh_{\widehat{\mathfrak{gl}}_{m+n}}$ wheel in color $(r, r, r+1)$ $1 \le r \le n-2$ implies vanishing as $q_1^{-\frac{m}{n}} \cdot \frac{z_{r,a}}{z_{r+1,b}} = qd, q_1^{\frac{m}{n}} \cdot \frac{z_{r+1,b}}{z_{r,c}} = qd^{-1}$, which means $\frac{z_{r,a}}{z_{r+1,b}} = qdq_1^{\frac{m}{n}} = \tilde{q}\tilde{d}$, $\frac{z_{r+1,b}}{z_{r,c}} = qd^{-1}q_1^{-\frac{m}{n}} = \tilde{q}\tilde{d}^{-1}$.

2. The $Sh_{\widehat{\mathfrak{gl}}_{m+n}}$ wheel in color $(r, r+1, r+1)$implies vanishing as $q_1^{\frac{m}{n}} \cdot \frac{z_{r+1,a}}{z_{r,b}} = qd^{-1}, q_1^{-\frac{m}{n}} \cdot \frac{z_{r,b}}{z_{r+1,c}} = qd$, which means $\frac{z_{r,b}}{z_{r+1,c}} = \tilde{q}\tilde{d}$, $\frac{z_{r+1,a}}{z_{r,b}} = \tilde{q}\tilde{d}^{-1}$.

3. Other cases can be verified similarly.

   Therefore, we have constructed a homomorphism $\Upsilon_{m+n,n}$.

$$\Upsilon_{m+n,n} : \bigoplus_{k_0,k_1,\ldots,k_{n-1}} Sh_{\widehat{\mathfrak{gl}}_{m+n}}(\underbrace{k_0,\ldots,k_0}_{m+1},\underbrace{k_1,\ldots,k_{n-1}}_{n-1})|_{q_1,q_2,q_3} \to$$
$$Sh_{\widehat{\mathfrak{gl}}_n}(\underbrace{k_0,\ldots,k_{n-1}}_{n})|_{\tilde{q}_1=q_1 q_1^{\frac{m}{n}},\tilde{q}_2=q_2,\tilde{q}_3=q_3 q_1^{-\frac{m}{n}}} \quad \text{given by}$$
$$\Upsilon_{m+n,n} : f\left(X_{0,1}^{0,k_0}, X_{1,1}^{1,k_0}, \ldots, X_{m,1}^{m,k_0}, X_{m+1,1}^{m+1,k_1}, \ldots, X_{m+n-1,1}^{m+n-1,k_{n-1}}\right) \to$$
$$f\left(Z_{0,1}^{0,k_0}, q_1^{-1} Z_{0,1}^{0,k_0}, \ldots, q_1^{-m} Z_{0,1}^{0,k_0}, q_1^{-m+\frac{m}{n}} Z_{1,1}^{1,k_1}, \ldots, q_1^{-\frac{m}{n}} Z_{n-1,1}^{n-1,k_{n-1}}\right) \cdot G$$

$$G = \frac{\prod\limits_{1\le i,j\le k_0} (z_{0,i} - qd^{-1} z_{0,j})^m \cdot \prod\limits_{a=0}^{n-1} \prod\limits_{1\le i\le k_a}^{1\le j\le k_{a+1}} (z_{a,i} - q_1^{\frac{m}{n}} z_{a+1,j})}{\prod\limits_{1\le i<j\le k_0} ((z_{0,i} - q^2 z_{0,j})^m \cdot (z_{0,i} - q^{-2} z_{0,j})^m)} \cdot \frac{\prod\limits_{a=1}^{n-1} \mu_a^{-k_a l_{a-1}} \cdot (-qd^2)^{\frac{k_0^2}{2}}}{\prod\limits_{a=0}^{n-1} \prod\limits_{1\le i\le k_a}^{1\le j\le k_{a+1}} (z_{a,i} - z_{a+1,j})}$$

To summarize, we have obtained that:

$$\tilde{d} = q^{-m} d^{m+1} \mu_1, \quad \tilde{d} = d\mu_2, \ldots, \tilde{d} = d\mu_{n-1}, \quad \tilde{d} = d \prod_{r=1}^{n-1} \mu_r^{-1}$$

Taking the product of the terms above, we have: $\tilde{d}^n = q^{-m} d^{m+n} = d(dq^{-1})^m$ and $\tilde{d} = d(dq^{-1})^{\frac{m}{n}} = dq_1^{\frac{m}{n}}$.
In particular,

$$\tilde{q}_1 = q_1 q_1^{\frac{m}{n}}, \quad \tilde{q}_2 = q_2, \quad \tilde{q}_3 = q_3 q_1^{-\frac{m}{n}}$$
$$\mu_2 = \mu_3 =, \ldots, = \mu_{n-1} = \tilde{d} d^{-1} = q_1^{\frac{m}{n}}$$
$$\mu_1 = \tilde{d} q^m d^{-(m+1)} = q_1^{-m+\frac{m}{n}}$$

Thus, overall we get 2^{n-1} such specialization homomorphisms when we start from color 0 variables. Allowing the extra rotation, we end up having precisely $n \cdot 2^{n-1}$ specialization homomorphisms.

4 Compatibility of subalgebra mappings

We conclude our paper with a few compatibility properties among realization maps and important subalgebras previously studied in [FHHSY], [FT]. Under the constructed homomorphism $\Upsilon_{m+n,n} : \bigoplus_{\underline{k}=(k_0\ldots k_m)} Sh_{\widehat{\mathfrak{gl}}_{m+n}} \to Sh_{\widehat{\mathfrak{gl}}_n}$, we explore the image and behavior of subalgebras under this mapping.

4.1 Bethe Subalgebras

Define Bethe subalgebra as a $\mathbb{N}$-graded subspace. Following [FHHSY] and Definition 1.8 of [FT], we introduce an important $\mathbb{Z}_+$-graded subspace $\mathcal{A}^{\mathrm{sm}} = \bigoplus_n \mathcal{A}_n^{\mathrm{sm}}$ of $Sh_{\mathfrak{gl}_n}$. Its degree n component is defined by

$$\mathcal{A}_n^{\mathrm{sm}} := \left\{ F \in Sh_{\mathfrak{gl}_n} \mid \partial^{(0;k)}F, \partial^{(\infty;k)}F \text{ exist and } \partial^{(0;k)}F = \partial^{(\infty;k)}F \quad 0 \leq k \leq n \right\}$$

$$\text{where } \partial^{(0;k)}F := \lim_{\xi\to 0} F\left(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n\right)$$
$$\partial^{(\infty;k)}F := \lim_{\xi\to \infty} F\left(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n\right)$$

With the particular specialization discussed above, we have:

Theorem 4.2 (Mapping Compatibility with Bethe Subalgebras)

$$\Upsilon_{m+n,n} : f\left(X_{0,1}^{0,N}, X_{1,1}^{1,N}, \ldots, X_{m,1}^{m,N}, X_{m+1,1}^{m+1,N}, \ldots, X_{m+n-1,1}^{m+n-1,N}\right) \to F\left(Z_{0,1}^{0,N}, q_3 Z_{0,1}^{0,N}, \ldots, q_3^m Z_{0,1}^{0,N}, q_3^{m-\frac{m}{n}} Z_{1,1}^{1,N}, \ldots, q_3^{\frac{m}{n}} Z_{n-1,1}^{n-1,N}\right) \cdot G$$

$$G = \frac{const \cdot \prod_{i,j=1}^{N} (z_{0,i} - q_3 z_{0,j})^m}{\prod_{1\leq i<j\leq N} \left((z_{0,i} - q^2 z_{0,j})(z_{0,i} - q^{-2} z_{0,j})\right)^m} \cdot \frac{\prod_{a=0}^{n-1}\prod_{i,j=1}^{N} (z_{a,i} - q_3^{-\frac{m}{n}} z_{a+1,j})}{\prod_{a=0}^{n-1}\prod_{i,j=1}^{N} (z_{a,i} - z_{a+1,j})}$$

and the new parameters are $\tilde{q}_1 = q_1 q_3^{-\frac{m}{n}}$, $\tilde{q}_2 = q_2, \tilde{q}_3 = q_3 q_3^{\frac{m}{n}}$

Proof.

Note that any segment $[a', b']$ rolled around $\mathbb{Z}/n\mathbb{Z}$ is arising from a segment $[a, b]$ rolled around $\mathbb{Z}/m + n\mathbb{Z}$.
Also recall the explicit generators of $A(\bar{s})$:

$$F_N^{\mu}(\bar{s}) = \frac{\prod_{i=0}^{m+n-1}\prod_{r\neq s\leq N}\left(x_{i,r} - q^{-2}x_{i,s}\right) \cdot \prod_{i=0}^{m+n-1}\left(s_0 \ldots s_i \prod_{r=1}^{N} x_{i,r} - \mu \cdot \prod_{r=1}^{N} x_{i+1,r}\right)}{\prod_{i=0}^{m+n-1}\prod_{r,s=1}^{N} (x_{i,r} - x_{i+1,s})}$$

Note that under above specialization:

$$\prod_{r=0}^{m+n-1} \prod_{r\neq s\leq N} \left(x_{i,r} - q^{-2}x_{i,s}\right) \to \prod_{i=0}^{n-1} \prod_{r\neq s\leq N} \left(z_{i,r} - q^{-2}z_{i,s}\right)$$

$$\prod_{i=0}^{m+n-1} \prod_{r,s=1}^{N} \left(x_{i,r} - x_{i+1,s}\right) \to \prod_{a=0}^{n-1} \prod_{r,s=1}^{N} \left(z_{a,r} - z_{a+1,s}\right)$$

Note however that the above factor in Υ has degree $m \cdot N$, hence, to preserve degree 0 component we can, for instance, add an element $\prod_{i=1}^{N} z_{0,i}^{-m}$ that is compatible with the shuffle product.
For $0 \leq i \leq m-1$, we have:
specialization of$\left\{ \left(s_0 \dots s_i \prod_{r=1}^{N} x_{i,r} - \mu \prod_{r=1}^{N} x_{i+1,r} \right) \right\} = const \cdot \prod_{r=1}^{N} z_{0,r}$
For $m \leq i \leq m+n-1$, we have:

$$\text{specialization of} \left\{ \left(s_0 \dots s_i \prod_{r=1}^{N} x_{i,r} - \mu \prod_{r=1}^{N} x_{i+1,r} \right) \right\} = const \cdot \left(s_0 \dots s_i \prod_{r=1}^{N} z_{i-m,r} - \mu \left(q_3^{-\frac{m}{n}} \right)^N \prod_{r=1}^{N} z_{i-m+1,r} \right)$$

Thus, we have

$$\left(s_0 \dots s_m \prod_{r=1}^{N} z_{0,r} - \mu \cdot q_3^{-\frac{mN}{r}} \prod_{r=1}^{N} z_{1,r} \right) \cdot \left(s_0 \dots s_m s_{m+1} \prod_{r=1}^{N} z_{1,r} - \mu \cdot q_3^{-\frac{mN}{n}} \prod_{r=1}^{N} z_{2,r} \right) \cdots \left(s_0 \dots s_{n+m-1} \prod_{r=1}^{N} z_{n-1,r} - \mu \cdot q_3^{\frac{-mN}{n}} \prod_{r=1}^{N} z_{0,r} \right)$$

Let $\tilde{\Upsilon} = \Upsilon \cdot \prod_{i=1}^{N} z_{0,i}^{-m}$, we have $\tilde{\Upsilon}_{m+n,n} : \bigoplus_{(k_0,\dots,k_m)} Sh_{\widehat{\mathfrak{gl}}_{m+n}} \to Sh_{\widehat{\mathfrak{gl}}_n}$.
Therefore, $\tilde{\Upsilon}\left(A\left(s_0, s_1, \dots, s_{m+n-1}\right)\right) = A\left(s_0, \dots, s_m, s_{m+1}, s_{m+n-1}\right) \subset Sh_{\widehat{\mathfrak{gl}}_n}$

4.3 horizontal Heisenberg

Theorem 4.4 *Heisenberg Subalgebra Compatibility*

$$\tilde{\Upsilon} : (Heis^{hor,+} \subseteq U(\widehat{\widehat{\mathfrak{gl}}}_{m+n})) = (Heis^{hor,+} \subseteq U(\widehat{\widehat{\mathfrak{gl}}}_n))$$

So, under $\tilde{\Upsilon}$, we also have $\tilde{\Upsilon} : (Heis^{hor,+}_{m+n}) = Heis^{hor,+}_n$.

Proof.

This result is a direct consequence of Theorem 4.2 in the limit $s_1, s_2, \dots s_{m+n-1} \to 0$ of [N3].

4.5 Subalgebra of slope 0

Following the definition provided in [N3], we recall slope 0 subalgebra, $A^{\circ}{}_{\widehat{\mathfrak{gl}}_{m+n}} = \{F \in Sh_{\widehat{\mathfrak{gl}}_{m+n}} \mid \lim_{\xi \to \infty} F_{\xi}^{\underline{l}} \quad 0 \le \underline{l} \le \underline{k}\}$

Let's check if this condition behaves well under the map of $\tilde{\Upsilon}$.

Proposition 4.6 *Compatibility of Subalgebra of slope 0*
$\widetilde{\Upsilon}$ *is compatible with subalgebras of slope 0, that is* $\Upsilon(F) \in A^{\circ}{}_{\widehat{\mathfrak{gl}}_n}$ *for* $\quad F \in A^{\circ}{}_{\widehat{\mathfrak{gl}}_{m+n}}$.

Proof.

Pick $\bar{l} = (l_0 \dots l_{n-1}) \preceq (k_0 \dots k_{n-1})$ then specializing $\{z_{i,r} | 0 \le i \le n-1, 1 \le r \le l_i\}$ to $\xi \cdot z_i^r$ with $\xi \to \infty$ in $\tilde{\Upsilon}(F)$ is equal to specialize $\{x_{i,r} | m+1 \le i \le m, 1 \le r \le l_0\} \cup \{x_{i,r} | m+1 \le i \le m+n-1, 1 \le r \le l_{i-m}\}$ to $\xi \cdot x_{i,r} \quad \xi \to \infty$.
We check the factor of t if it has limit as $\xi \to \infty$.

$$t = \frac{\prod_{r,s=1}^{k_0} (z_{0,r} - q_3 z_{0,s})^m \cdot \prod_{a=0}^{n-1} \prod_{1 \le r \le k_a}^{1 \le s \le k_{a+1}} (z_{a,r} - q_3^{-\frac{m}{n}} z_{a+1,s})}{\prod_{a=0}^{n-1} \prod_{1 \le r \le k_a}^{1 \le s \le k_{a+1}} (z_{a,r} - z_{a+1,s}) \cdot \prod_{1 \le r \ne s \le k_0} (z_{0,r} - q^2 z_{0,s})^m \cdot \prod_{r=1}^{k_0} z_{0,r}^m}$$

As $\xi \to \infty$, the numerator of t approaches

$$m(k_0^2 - (k_0 - l_0)^2) + \sum_{a=0}^{n-1} (k_a k_{a+1} - (k_a - l_a)(k_{a+1} - l_{a+1}))$$

the denominator of t approaches

$$m(k_0(k_0-1) - (k_0-l_0)(k_0-l_0-1)) + ml_0 + \sum_{a=0}^{n-1} (k_a k_{a+1} - (k_a - l_a)(k_{a+1} - l_{a+1}))$$

Note that

$$k_0^2 - (k_0 - l_0)^2 = 2k_0 l_0 - l_0^2$$
$$k_0(k_0-1) - (k_0 - l_0)(k_0 - l_0 - 1) + l_0 = 2k_0 l_0 - l_0^2$$

Therefore, the factor of t has nonzero limit as we specialize $z_{i,r}$ to $\xi \cdot z_{i,r}$ when $\xi \to \infty$. Under $\tilde{\Upsilon}$, we have $\tilde{\Upsilon}(A_{\widehat{\mathfrak{gl}}_{m+n}}^{slope=0}) \subseteq A_{\widehat{\mathfrak{gl}}_n}^{slope=0}$, which means $\tilde{\Upsilon}$ is surjective.

5 Further directions

From our discussion of mappings between $Sh_{\widehat{\mathfrak{gl}}_{m+n}} \to Sh_{\widehat{\mathfrak{gl}}_n}$, two questions arise for further explorations: From our discussion of algebra homomorphisms connecting subalgebras of $Sh_{\widehat{lgl_{m+n}}}$ with $Sh_{\widehat{lgl_n}}$, two types:

1. Besides for subspaces/subalgebras $Sh(k_0 \dots k_0, k_1 \dots k_1)$, there are many more subalgebras coming from constants on degrees.
2. It's a very interesting question to generalize the present result of Υ to the $Sh_{\widehat{\mathfrak{g}}}$ for $\mathfrak{g}$ simple Lie algebras of other types, in particular, to types B_n, C_n, D_n.

Acknowledgements

My deepest thanks to my mentor, Professor Alexander Tsymbaliuk, for his kindness, guidance, and patience with the tiniest details. I am thankful for his support and he helped me realize both the hardship and fun that comes with writing a paper. I would also like to thank my family, who created the environment in which countless hours of work were done in.

References

[BT] M. Bershtein, A Tsymbaliuk, *Homomorphisms between different quantum toroidal and affine Yangian algebras*, JPAA 223 (2019), no. 2, 867–899

[DF] J. Ding, I. Frenkel, *Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$*, Comm. Math. Phys. **156** (1993), no. 2, 277–300.

[FJMM] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Branching rules for quantum toroidal $\mathfrak{gl}_\mathfrak{n}$*, Adv. Math. **300** (2016), 229–274.

[FHHSY] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida, A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials, J.Math.Phys., 50(9):0952515, 42,2009.

[FO] B. Feigin, A. Odesskii, *Elliptic deformations of current algebras and their representations by difference operators*, (Russian) Funktsional. Anal. i Prilozhen. **31** (1997), no. 3, 57–70; translation in Funct. Anal. Appl. **31** (1998), no. 3, 193–203.

[FT] B. Feigin, A. Tsymbaliuk, *Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$ via shuffle algebras*, Selecta Math. (N. S.) **22** (2016), no. 2, 979–1011.

[N1] A. Neguţ, *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN (2014), no. 22, 6242–6275.

[N2] A. Neguţ, *Quantum toroidal and shuffle algebras*, Adv. Math. **372** (2020), Paper No. 107288.

[N3] A. Neguţ, *The PBW Basis of* $U_{q,\bar{q}}(\ddot{gl}_n), Transformation Groups (2022), DOI: 10.1007/s00031-022-09696-x.pdf.$

[NT] A. Neguţ, A. Tsymbaliuk, *Quantum loop groups and shuffle algebras via lyndon words*, https://arxiv.org/pdf/2102.11269.pdf

[T1] A. Tsymbaliuk, *PBWD bases and shuffle algebra realizations for* $U_v(\mathcal{L}\mathfrak{sl}_n), U_{v1,v2}(\mathcal{L}\mathfrak{sl}_n), Uv(\mathcal{L}\mathfrak{sl}(m|n))$ *and their integral forms*, Selecta Math. (N.S.) 27 (2021), no. 3, Paper No. 35, 48pp