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Introduction, by Yehor Avdieiev

In this paper, we explore some fascinating applications of the well-known even for young mathematicians method of the *Inclusion-Exclusion*. This is one of the most fundamental tools in enumerative combinatorics. The goal of the method is to determine the cardinality of a set Q by approximating the answer with an overcount, then subtracting off or adding on an overcounted approximation of our original error, and so on, until we "converge" to the correct answer after finitely many steps. One of the simplest examples of using the Principle of Inclusion-Exclusion is:

Example 0.1. *How many integers $1 \leq n \leq 1000$ are not divisible by 3, 5 and 7?*

Proof. First, start from 1000. Then substract separately amount of numbers that are divisible by 3, or 5, or 7. We will get $1000-[\frac{1000}{3}]-[\frac{1000}{5}]-[\frac{1000}{7}]$. We took away more than once numbers that are divisible by at least two of those three integers. Let's add amount of numbers that are divisible by 15, or by 21, or by 35. We will get $1000-[\frac{1000}{3}]-[\frac{1000}{5}]-[\frac{1000}{7}]+[\frac{1000}{15}]+[\frac{1000}{21}]+[\frac{1000}{35}]$. We overcounted numbers that are divisible by all of the three integers. It's easy to understand that the final answer will be $1000-[\frac{1000}{3}]-[\frac{1000}{5}]-[\frac{1000}{7}]+[\frac{1000}{15}]+[\frac{1000}{21}]+[\frac{1000}{35}]-[\frac{1000}{105}]=457$. $\square$

It turns out that the *Principle of Inclusion-Exclusion* applies in much more complex combinatorial problems. We will now describe the basic formula of this principle. Let A be a finite set of objects and S be a set of properties that the elements of A may satisfy or not. We define by $f_{=}(Y)$ (respectively, $f_{\geq}(Y)$) the number of objects in a set A having exactly (respectively, at least) properties in $Y \subseteq S$. Then the inclusion-exclusion formula may be stated as

$$f_{=}(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f_{\geq}(Y) \tag{1}$$

for any subset T $\subseteq S$. For $T=\emptyset$, this reads as $f_{=}(\emptyset)=\sum_{Y\subseteq S}(-1)^{|Y|}f_{\geq}(Y)$ (while the general case can be easily reduced to that one by replacing S by $S\backslash T$ and X by $f_{\geq}(Y) \subseteq X$). Moving all the negative terms to the other side, we get an equivalent formula:

$$f_{=}(\emptyset) + \sum_{|Y|-odd} f_{\geq}(Y) = \sum_{|Y|-even} f_{\geq}(Y). \tag{2}$$

Once two sets are known to have the same cardinality, it is natural to seek a bijection between them. Let us prove the formula (2) using this idea.

Proof. The right-hand side of the equation (2) is the cardinality of the set N of triples $(x;Y;Z)$, where $x \in A$ has exactly properties Z and $Y \subseteq Z$ is a subset of even cardinality $|Y|$. The left-hand side of (1) is the cardinality of the set $N' \cup M'$, where N' consists of triples $(x';Y';Z')$, where $x' \in A$ posseses exactly properties Z' and $Y' \subseteq Z'$ is a subset of odd cardinality $|Y'|$, while the set M' consists of those $x \in A$ that satisfy none of the properties in S. Choose any total ordering on the set S of properties. This allows us to define $min(Y)$, $min(Z)$ for any $Y \subseteq Z \subseteq S$. We also set $min(Y) = \infty \; if \; Y = \emptyset$. Let's define two maps $F\colon M' \cup N' \to N$ and $G\colon N \to M' \cup N'$ via

$$F : \left[\begin{array}{l} x \mapsto (x;\emptyset;\emptyset) \\ (x';Y';Z') \mapsto (x';Y' \cup i;Z'),\; if\; i = min(Z') < min(Y') \\ (x';Y';Z') \mapsto (x';Y' - i;Z'),\; if\; i = min(Z') = min(Y') \end{array} \right.$$

and

$$G : \left[\begin{array}{l} (x;\emptyset;\emptyset) \mapsto x \\ (x;Y;Z) \mapsto (x;Y \cup i;Z),\; if\; i = min(Z) < min(Y) \\ (x;Y;Z) \mapsto (x;Y - i;Z),\; if\; i = min(Z) = min(Y) \end{array} \right.$$

It's obvious that $F \circ G = Id_N$ and $G \circ F = Id_{M' \cup N'}$, hence, F and G are inverse bijective maps. This establishes a natural combinatorial proof of the formula (2). $\square$

1 Permutations with Restricted Position and Rook's problems, by Kseniya Drozdova

Let $\omega \in \mathfrak{S}_n$ be a permutation of the set $[n] = \{1, 2, \ldots, n\}$. Then one may be interested in the problems where we need to find number of ω such that some values for $\omega(i)$ are not allowed. As an example, we shall compute in Example 1.2 the number of permutations $\omega \in \mathfrak{S}_n$ such that $\omega(i) \not\equiv i (\mathrm{mod}\; n)$ for $i = \overline{1,n}$. We shall develop some general theory that allows to solve many of such problems.

Any subset B of $[n] \times [n]$ is called a *board*. For ω as above its *graph* $G(\omega)$ is defined via

$$G(\omega) = \{(i, \omega(i)) : \; i \in [n]\}.$$

Then, we define

$$N_j = \#\{\omega \in \mathfrak{S}_n : j = \#(B \cap G(\omega))\}$$

and consider the corresponding polynomial (the generating function of all N_j):

$$N_n(x) := \sum_{j=0}^{n} N_j x^j.$$

We also define r_k as the number of placing k non-attacking rooks on B, which is the same as the number of k-subsets of B such that no two elements have a common coordinate. The corresponding generating function $r_B(x)$ is called a *rook polynomial of the board* B:

$$r_B(x) = \sum_k r_k x^k.$$

The following result establishes a relation between N_j's and r_k's.

Theorem 1.1. *We have*

$$N_n(x) = \sum_{k=0}^{n} r_k (n-k)!(x-1)^k. \tag{3}$$

First proof. The left-hand side counts the number of placing n non-attacking rooks on $[n]\times[n]$ and labeling those on B in one of x colours $\{1, 2, \dots, x\}$. The right-hand side counts the number of ways to place k non-attacking rooks on B and labeling them in one of the $x-1$ colours (without colour "1"), and then placing the remaining $n-k$ rooks in $(n-k)!$ ways, as they should occupy $n-k$ empty rows and columns, and painting those that get placed on B into the colour "1". Clearly, there is a bijection between these sets[1], hence the equality (3). □

Second proof. For $0 \le k \le n$, let C_k denote the number of pairs (C, ω), where ω is a permutation of $[n]$ and C is a k-element subset of $B \cap G(\omega)$. For every $j \ge k$ there are N_j ways to choose ω so that $j = \#(B \cap G(\omega))$, then C can be chosen in $\binom{j}{k}$ ways. Therefore $C_k = \sum_{j=k}^{n} N_j \binom{j}{k}$. But on the other hand, we could start with choosing C in r_k ways and then extending it to ω in $(n-k)!$ ways (as explained above), so $C_k = r_k(n-k)!$. Therefore $r_k(n-k)! = \sum_j N_j \binom{j}{k}$. Multiplying this by $(x-1)^k$ and summing over all $0 \le k \le n$ we get

$$\sum_k r_k(n-k)!(x-1)^k = \sum_{n\ge j\ge k\ge 0} N_j \binom{j}{k}(x-1)^k = \sum_{j=0}^{n} N_j \sum_{k=0}^{j} \binom{j}{k}(x-1)^k = \sum_{j=0}^{n} N_j x^j = N_n(x),$$

with the third equality due to the binomial theorem.

□

Comparing the coefficients of x^0 in both sides of the equality (3), we immediately obtain:

Corollary 1.1.1.

$$N_0 = N_n(0) = \sum_{k=0}^{n} r_k(n-k)!(-1)^k.$$

We can also prove Corollary 1.1.1 by a direct application of the Inclusion-Exclusion principle:

Proof. It easy to see that, this equality is true:

$$\sum_X f_=(X) x^{\#X} = \sum_Y f_\ge(Y)(x-1)^{\#Y}$$

by using inclusion-exclusion formula $f_=(X) = \sum_{Y\supseteq X} f_\ge(Y)(-1)^{\#(Y-X)}$ and comparing the corresponding coefficients. Think of the condition that there are k rooks on the board B as the k-th property of B. Therefore $\sum_{\#X=j} f_=(X) = N_j$ and $\sum_{\#Y\ge j} f_=(X) = r_j(n-k)!$. Setting $x=0$, we get Q.E.D. □

[1] Let X be a set of left-hand side colorings with painting all rooks that don't stand on B in color 1 and Y be a set of right-hand side colorings with removing color 1. Then we have a bijection map $f : X \to Y$ given as follows: for $x \in X$ we get $f(x)$ by removing colours from all the rooks in x that are painted in color 1. Inverse map $f^{-1} : Y \to X$ is given by: for $y \in Y$ we get $f^{-1}(y)$ by coloring all the rooks that don't have a colour in colour 1.

Let us illustrate the above notions and results in the following particular setup:

Example 1.2. *Find the number of permutations $\omega \in \mathfrak{S}_n$ such that $\omega(i) \not\equiv i (mod\ n)$ for $i = \overline{1, n}$.*

Solution. Let $B = \{(1,1); (2,2); \ldots; (n,n)\}$. As no two squares of B share the common coordinate, any placement of rooks on B is non-attacking, hence, $r_k = \binom{n}{k}$. Then, according to Corollary 1.1.1 the desired number N_0 equals:

$$N_0 = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(-1)^k = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Problème des ménages

This problem is asking for the number $M(n)$ of permutations $\omega \in S_n$ such that $\omega(i) \not\equiv i, i+1 \pmod{n}$ for all $i \in [n]$. In other words, we seek N_0 for the board

$$B = \{(1,1); (2,2); \ldots; (n,n); (1,2); (2,3); \ldots; (n,1)\}$$

We note right away that r_k for this board is the number of ways to choose k points on a circle with $2n$ points, such that no two are consecutive.

Theorem 1.3. *The number of ways to choose k points, no two consecutive, from a collection of m points arranged in a circle is $\frac{m}{m-k}\binom{m-k}{k}$.*

First proof. Let's label all points from 1 to m in clockwise order. There are $m-k$ not chosen points, and between any two consecutive of those there may be no more then one chosen point, so we should choose k spaces from $m-k$ places. So, if point 1 is not chosen then there are $m-k$ places for k selected points, so the number of different ways you can choose k things from a collection of $m-k$ of them is $\binom{m-k}{k}$.
If the point 1 will be among the chosen k points, then there are $m-k-1$ free spaces because one place is already taken and $k-1$ more chosen points. This amounts to choosing $k-1$ objects from a collection of $m-k-1$, and the number of such choices is $\binom{m-k-1}{k-1}$. Therefore, the desired number is $\binom{m-k}{k} + \binom{m-k-1}{k-1} = \binom{m-k}{k} + \frac{k}{m-k}\binom{m-k}{k} = \frac{m}{m-k}\binom{m-k}{k}$. □

Second proof. Let $f(m,k)$ be the desired number and let $g(m,k)$ be the number of ways to color in red k points, such that any two of them are nonconsecutive, from m points arranged in a circle, and then coloring one of the non-red points blue. Let's start by choosing a blue point, there are m options to do this. The number of non-red points is $m-k$, inserting k red points between them, we get that there are no two consecutive red points, in total there are $\binom{m-k}{k}$ such choices. Thus, $g(m,k) = m\binom{m-k}{k}$. But obviously $g(m,k) = (m-k)f(m,k)$. Therefore $f(m,k) = \frac{m}{m-k}\binom{m-k}{k}$. □

Combining Theorems 1.1 and 1.3, we obtain:

Corollary 1.3.1. *The polynomial $N_n(x)$ for the board $B = \{(i,i), (i,i+1) (mod\ n) : 1 \leq i \leq n\}$ is given by*

$$N_n(x) = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!(x-1)^k.$$

In particular,

$$N_0 = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!(-1)^k,$$

where N_0 is number of the permutations $\omega \in S_n$ such that $\omega(i) \not\equiv i, i+1$ (mod n) for $1 \le i \le n$.

We shall now consider a special class of boards.

Ferrers boards

A Ferrers board of shape $(b_1, b_2, b_3, \ldots, b_m)$, where $b_i \in \mathbb{N}$ for $i \in [n]$ and $b_1 \le b_2 \le b_3 \le \cdots \le b_m$ is defined by:

$$B = \{(i,j) : 1 \le i \le m, 1 \le j \le b_i\}$$

so B is obtained from the Young diagram of the partition $\lambda = (b_m, \ldots, b_1)$ by first reflecting in x-axis, then rotating 90° counterclockwise, and then parallelly transporting by the vector $(m, 0)$.

For $n \in \mathbb{N}$ we define the polynomial $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$.

Theorem 1.4. *Let $\sum r_k x^k$ be the rook polynomial of the Ferrers board B of shape $(b_1, \ldots, b_m)$. Set $s_i = b_i - i + 1$. Then, we have:*

$$\sum_k r_k \cdot (x)_{m-k} = \prod_{i=1}^{m} (x + s_i). \tag{4}$$

Proof. To verify an equality of polynomials it suffices to check their equality at all $x \in \mathbb{N}$, because if two polynomials have the same values at an infinite number of points then they coincide. To this end, we consider the board $B' = B \cup C$ where C is a rectangle $x \times m$ placed right below B. We shall now count $r_m(B')$ in two ways:

1. Since B' has m columns, each must contain exactly 1 rook. There are $x + b_1 = x + s_1$ choices to place a rook in the first column. After doing so, there are $x + b_2 - 1 = x + s_2$ choices to place a rook in the second column, ... Thus, overall we get $r_m(B') = (x + s_1) \ldots (x + s_m)$.

2. On the other hand, we can count separately placements of rooks on B and C. For every k, there are r_k ways to place k non-attacking rooks on B. We claim that the remaining $m - k$ rooks may be placed on C in $(x)_{m-k}$ ways, because there are x choices to place a rook in the leftmost yet unoccupied column, after that there are $x-1$ choices to place a rook in the leftmost yet unoccupied column, ..., there $x - m + k + 1$ ways to place a rook in the last unoccupied column.

This completes our proof since both sides of (4) count $r_m(B')$. $\square$

Let $S(n,k)$ be the *Stirling number of the second kind*, that is the number of partitions of an n-element set into k nonempty sets.

Corollary 1.4.1. *Let B be the triangular board (or staircase) of shape $(0, 1, 2, \ldots, m-1)$. Then $r_k = S(m, m-k)$.*

Proof. To begin, we shall first prove

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k \tag{5}$$

Let X be a set of x different elements and N be a set of n different elements. Then it is clear that the left-hand side of (5) counts the number of all functions $f\colon N \to X$. Each function is surjective onto a subset Y of X, where $Y = \{f(a) : a \in N\}$. Let $k = \#Y$, then for every $k = \overline{1,x}$ the number of such functions is $S(n,k)(x)_k$, since there are $S(n,k)$ ways to divide N into k nonempty subsets, for the first subset from N there are x ways to choose their common values in X, for the second there are $x-1$ choices, ... , for the last subset there are $x-k+1$ choices. Hence, (5) is proven.

Now we are ready to prove the Corollary. Since $s_i = 0\ \forall\ i \in [m]$ for the triangular board B, we have $x^m = \sum_{k=0}^{m} r_k(x)_{m-k}$. Comparing with (5), we get $\sum_{k=0}^{m} r_k(x)_{m-k} = \sum_{k=0}^{m} S(m, m-k)(x)_{m-k}$. But $\{(x)_{m-k} \mid 0 \le k \le m\}$ are obviously linearly independent (as they have pairwise distinct degrees), hence, $r_k = S(m, m-k)$. □

Corollary 1.4.2. *Two Ferrers boards, each with m columns (allowing empty columns), have the same rook polynomial if and only if the corresponding multisets of the numbers s_i are the same.*

Proof. Follows immediately from (4). □

Theorem 1.5. *Let $0 \le c_1 \le \ldots \le c_m$, and let $f(c_1, \ldots, c_m)$ be the number of Ferrers boards with no empty columns and having the same rook polynomial as the Ferrers board of shape $(c_1, \ldots, c_m)$. Add enough initial $0'$s to $c_1, \ldots, c_m$ to get a shape $(b_1, \ldots, b_t) = (0, 0, 0, \ldots 0, c_1, \ldots, c_m)$ such that if $s_i = b_i - i + 1$ then $s_1 = 0$ and $s_i < 0$ for $i = \overline{2,t}$. Suppose that a_i of s_j's are equal to $-i$, so in particular, $\sum_{i \ge 1} a_i = t-1$. Then, we have:*

$$f(c_1, c_2, \ldots, c_m) = \binom{a_1 + a_2 - 1}{a_2}\binom{a_2 + a_3 - 1}{a_3}\binom{a_3 + a_4 - 1}{a_4}\ldots$$

Proof. We know that $s_i - 1 \le s_{i+1}$, but $s_i \le 0$, so by Corollary 1.4.2, we must count the number of permutations of the multiset $\{1^{a_1}, 2^{a_2}, 3^{a_3}, \ldots\}$ such that $d_i + 1 \ge d_{i+1}$ for $i = \overline{1, t-2}$ (with the corresponding Ferrers board being $-d_1,\ -d_2+1,\ -d_3+2,\ \ldots$). Place a_1 $1'$s in a line. "2"'s can go only after "2" or "1", therefore we can place few "2" after each "1" in a_1 space so there are $\left(\!\binom{a_1}{a_2}\!\right) = \binom{a_1+a_2-1}{a_2}$ ways. Analogously there are $\binom{a_2+a_3-1}{a_3}$ ways to place "3" (we can't insert "3" after "1") etc. This completes the proof. □

For two boards A and B: $A \subseteq [n] \times [m]$ and $B \subseteq [p] \times [q]$[2], we define their union $A \sqcup B$ as a subset of $[n+p] \times [m+q]$ given by:

$$A \sqcup B = \{(i,j) : (i,j) \in A \text{ or } (i-n, j-m) \in B\}.$$

Then, we have the following simple property:

Lemma 1.6. $r_{A \sqcup B}(x) = r_A(x) r_B(x)$

[2] This notion of a board does not really contradict the earlier one used to denote a subset of a square $[n] \times [n]$

Proof. Since A and B do not have common rows and columns placing j non-attacking rooks on $A \sqcup B$ amounts to placing k non-attacking rooks on A and $j-k$ non-attacking rooks on B for some $0 \leq k \leq j$. Hence $r_j^{(A \sqcup B)} = \sum_{k=0}^{j} r_k^{(A)} r_{j-k}^{(B)}$ (with a superscript denoting the board under consideration), and thus $r_{A \sqcup B}(x) = r_A(x) r_B(x)$. $\square$

We conclude this section by illustrating how the above machinery can be applied to real-life problems.

Consider the following problem. There are 6 tickets for different amusement rides: Ferris Wheel (F), Tilt-A-Whirl (T), Insanity (I), Scrambler (S), Rotor (R) and Bumper Cars (B) and 6 children: Amelia (A), Charley (C), Felix (F), Gabriel (G), Sophie (S) and Kate (K) want to distribute them among themselves. We know that:

a. Amelia doesn't like Insanity and Rotor
b. Charley doesn't like Ferris Wheel
c. Felix doesn't like Tilt-A-Whirl
d. Gabriel doesn't like Scrambler and Bumper Cars
e. Sophie doesn't like Ferris Wheel and Rotor
f. Kate doesn't like Insanity

In how many ways can they distribute tickets among themselves so that everybody will be satisfied?

Solution

Note on the chessboard kind of amusement rides that children don't like. Then, we get (see Fig.1):

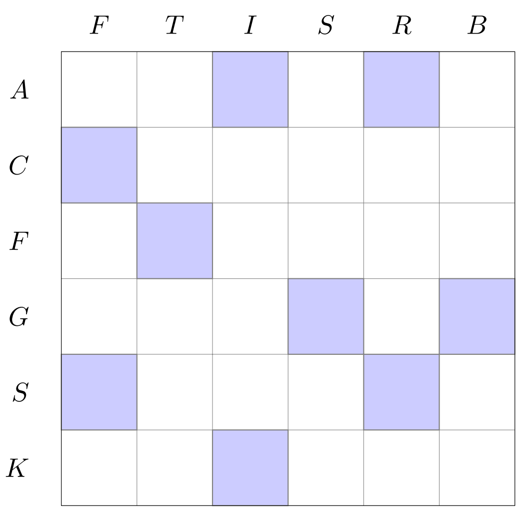

Fig.1

If we rearrange some rows and columns as shown on Fig.2, then the board B consisting of the blue-colored boxes splits as a dijoint union $B = B_1 \sqcup B_2$ with B_1 consisting of blue boxes in the rectangle $IRF \times SKAC$ and B_2 consisting of blue boxes in the rectangle $TSB \times FG$.

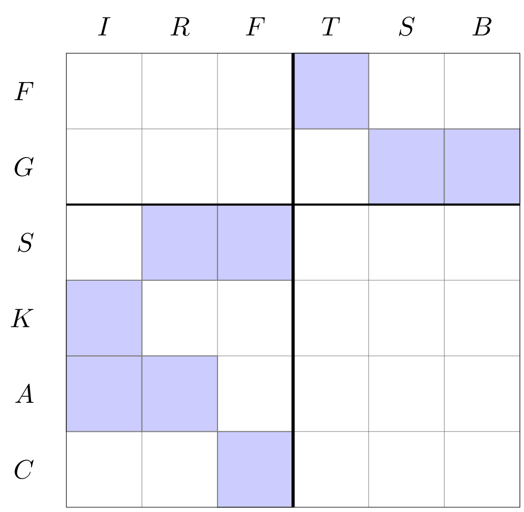

Fig.2

According to Lemma 1.6, we have: $r_B(x) = r_{B_1}(x)r_{B_2}(x)$.

On B_1 there is a unique way to place 0 rooks, 6 ways to place 1 rook, 10 ways to place 2 rooks ($R \times S - I \times K,\ R \times S - I \times A,\ R \times S - F \times C,\ F \times S - I \times K,\ F \times S - I \times A,\ F \times S - R \times\ A,\ I \times K - R \times A,\ I \times K - F \times C,\ I \times A - F \times C,\ R \times A - F \times C$) and 4 ways to place 3 rooks, and 0 ways to place more than 3 rooks, so $r_{B_1}(x) = 1 + 6x + 10x^2 + 4x^3$. Similarly, we get $r_{B_2}(x) = 1 + 3x + 2x^2$. Therefore, it easy to see that $r_B(x) = r_{B_1}(x)r_{B_2}(x) = (1 + 6x + 10x^2 + 4x^3)(1 + 3x + 2x^2) = 1 + 9x + 30x^2 + 46x^3 + 32x^4 + 8x^5$. Then, by Corollary 1.1.1, the desired number is

$$N_0 = 6! - 9 \cdot 5! + 30 \cdot 4! - 46 \cdot 3! + 32 \cdot 2! - 8 \cdot 1! = 140.$$

This means there are 140 ways to distribute tickets.

2 V-partitions and Unimodal Sequences, by Yehor Avdieiev

We shall now present an example of a sieve method which is similar to (but is not implied by) the principle of inclusion-exclusion. We define by a *unimodal sequence of weight* n a sequence of positive integers $a_1a_2a_3\cdots a_m$ that satisfies:

a. $\sum a_i = n$

b. $\exists\ j:\ \mathrm{a}_1 \leq a_2 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_m$

We can find a generating function $U(q)$ of the total number of unimodal sequences of weight n. Let $u(n)$ denote the total number of unimodal sequences of weight n with $u(0)$=0, and consider the corresponding generating function $U(q) = \sum_{n\geq 0} u(n)q^n$. Our goal is to find an explicit formula for $U(q)$. To this end, we define $[k]! := (1-q)(1-q^2)\cdots(1-q^k)$.

Lemma 2.1. *U(q)*=$\sum_{k\geq 1} \frac{q^k}{[k]![k-1]!}$

Proof. It is clear that every unimodal sequence with the largest term k has the form:

$$w = a_1a_2\cdots a_m$$

$$= \underbrace{11\cdots 1}_{b_1}\underbrace{22\cdots 2}_{b_2}\cdots\underbrace{kk\cdots k}_{b_k}\underbrace{(k-1)\cdots(k-1)}_{c_{k-1}}\cdots\underbrace{11\cdots 1}_{c_1}$$

for some $b_1,\ldots,b_{k-1},c_1,\ldots,c_{k-1}\geq 0$ and $b_k\geq 1$.

On the other hand, we can rewrite $\frac{q^k}{[k]![k-1]!}$ in the form:

$$\frac{q^k}{[k]![k-1]!} =$$
$$(1+q+q^2+\ldots)\cdots(1+q^{k-1}+q^{2(k-1)}+\ldots)(q^k+q^{2k}+q^{3k}+\ldots)(1+q^{k-1}+\ldots)\cdots(1+q+q^2+\ldots)$$

Having b_1 of $1's$ in w will cause to choose $q^{1\cdot b_1}$ from the first bracket, having b_2 of $2's$ in w will cause to choose $q^{2\cdot b_2}$ from the second bracket and so on. We note that the free term 1 is missing in $(q^k+q^{2k}+\ldots)$ due to the restriction $b_k\geq 1$. □

As often happens in combinatorics, infinite sums may be written as infinite products. Our main goal is to obtain such a product formula for $U(q)$. To this end, we shall work with objects slightly different from unimodal sequences and then connect them with unimodal sequences. We define a *V-partition of* n to be an array:

$$\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$$

such that all numbers are natural, $c+\sum a_i+\sum b_j=n$, $c\geq a_1\geq a_2\geq\cdots$, $c\geq b_1\geq b_2\geq\cdots$. It's easy to see, that a *V-partition* is a unimodal sequence of the same weight, but with a rooted largest element. Let's define $\upsilon(n)$ to be the number of a *V-partitions of weight* n and set $\upsilon(0)=1$. So:

Lemma 2.2.

$$V(q):=\sum_{n\geq 0}\upsilon(n)q^n=\sum_{k\geq 0}\frac{q^k}{[k]![k]!}$$

Proof. Proof of this fact is straightforward and is completely analogous to that of the Lemma 2.1. Let us just note that we presently have two brackets $(1+q^k+q^{2k}+\ldots)$, because we should root the largest element in the *V-partition*. □

Let us introduce yet another actor, the set D_n of double partitions of n:

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix}$$

such that $a_i,\ b_j\in\mathbb{N}$ and $\sum a_i+\sum b_j=n$, $a_1\geq a_2\geq\ldots$, $b_1\geq b_2\geq\cdots$. Let's set $d(n)=|D_n|$, with the convention $d(0)=1$. The following result is obvious:

Lemma 2.3.

$$D(q):=\sum_{n\geq 0}d(n)q^n=\prod_{k\geq 1}(1-q^k)^{-2}$$

Let V_n be the set of *V-partitions of* n, so that $|V_n| = v(n)$. Define a map $F_1\colon D_n \to V_n$ by

$$F_1 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, \ \mathit{if}\ a_1 \geq b_1 \\ \begin{bmatrix} & a_1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, \ \mathit{if}\ b_1 > a_1 \end{cases}$$

We count every *V-partition* that has a form $\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$ with $c > a_1$ twice, because it is the image of both $\begin{bmatrix} c & a_1 & a_2 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ c & b_1 & b_2 & \cdots \end{bmatrix}$, while all other *V-partitions* are counted precisely once. Let V_n^1 be the set of the former *V-partitions*, i.e. those with $c > a_1$. By above, we get:

$$|V_n| = |D_n| - |V_n^1|.$$

Now we can define a new map $F_2\colon D_{n-1} \to V_n^1$ by

$$F_2 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 + 1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, \ \mathit{if}\ a_1 + 1 \geq b_1 \\ \begin{bmatrix} & a_1 + 1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, \ \mathit{if}\ b_1 > a_1 + 1 \end{cases}$$

It's easy to understand that we count every *V-partition* that has a form $\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$ with $c > a_1 > a_2$ twice, because it arises as image of both $\begin{bmatrix} a_1 - 1 & a_2 & a_3 & \cdots \\ c & b_1 & b_2 & \cdots \end{bmatrix}$ and $\begin{bmatrix} c - 1 & a_1 & a_2 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{bmatrix}$, while all other *V-partitions* in V_n^1 are counted precisely once. Let's name the former set of *V-partitions* by V_n^2. Then, we get:

$$|V_n| = |D_n| - |V_n^1| = |D_n| - \big(|D_{n-1}| - |V_n^2|\big) = |D_n| - |D_{n-1}| + |V_n^2|.$$

Then define the next map $F_3\colon D_{n-3} \to V_n^2$ by

$$F_3 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 + 1 & a_3 & \cdots \\ a_1 + 2 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, \ \mathit{if}\ a_1 + 2 \geq b_1 \\ \begin{bmatrix} & a_1 + 2 & a_2 + 1 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, \ \mathit{if}\ b_1 > a_1 + 2 \end{cases}$$

Let's denote a subset of V_n^2 with $c > a_1 > a_2 > a_3$ by V_n^3. Arguing as above, we get:

$$|V_n| = |D_n| - |D_{n-1}| + |D_{n-3}| - |V_n^3|.$$

By continuing this process, we obtain the following formula:

$$|V_n| = \sum_{i\geq 1}(-1)^{i-1}\left|D\left(n-\binom{i}{2}\right)\right| = \sum_{i\geq 1}(-1)^{i-1}d\left(n-\binom{i}{2}\right) \tag{6}$$

From the formula (6) and Lemma 2.3 we get the following result:

$$V(q) = \sum_{n\geq 0}\left((-1)^n q^{\frac{n(n+1)}{2}}\right)D(q) = \sum_{n\geq 0}\left((-1)^n q^{\frac{n(n+1)}{2}}\right)\prod_{k\geq 1}(1-q^k)^{-2}. \tag{7}$$

We also have the following simple result connecting all three generating functions introduced above:

Lemma 2.4. $U(q) + V(q) = D(q) = \prod_{k\geq 1}(1-q^k)^{-2}$

Proof. We need to find a bijection $D_n \leftrightarrow U_n \cup V_n$. Such a bijection is given by:

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, \; if\ a_1 \geq b_1 \\ \cdots a_2 a_1 b_1 b_2 \cdots, \; if\ b_1 > a_1 \end{cases}$$

$\square$

Now we can finally derive the product type formula for $U(q)$:

Theorem 2.5. $U(q)$=$\prod_{k\geq 1}(1-q^k)^{-2}\sum_{n\geq 1}\left((-1)^{n-1}q^{\frac{n(n+1)}{2}}\right)$

Proof. Combining Lemma 2.4 and equation (7), we obtain:

$$U(q) = D(q) - V(q) = \prod_{k\geq 1}(1-q^k)^{-2} - \sum_{n\geq 0}\left((-1)^n q^{\frac{n(n+1)}{2}}\right)\prod_{k\geq 1}(1-q^k)^{-2} =$$
$$\prod_{k\geq 1}(1-q^k)^{-2}\sum_{n\geq 0}\left((-1)^{n-1}q^{\frac{n(n+1)}{2}}\right).$$

$\square$

Many famous sequences in mathematics are known to be unimodal, see [1, Exercise 1.50]. A sequence of real numbers $a_1, a_2, \cdots, a_n$ is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$.

Example 2.6. *If a sequence of positive real numbers $a_1, a_2, \cdots, a_n$ is log-concave then $a_1, a_2, \cdots, a_n$ is unimodal.*

Proof. Suppose for contradiction that the sequence is not unimodal. Thus $\exists\ i, j$: $a_i > a_{i+1} \geq a_{i+2} \geq \cdots \geq a_j < a_{j+1}$. But then $a_j^2 < a_{j+1}a_{j-1}$, contradicting the log-concave assumption. $\square$

A polynomial $F(x) = \sum_{i=0}^{n} a_i x^i$ with real coefficients is called *unimodal* (respectively, *symmetric* with center $\frac{n}{2}$) if $a_0, a_1, \ldots, a_n$ is a unimodal (respectively, symmetric with center $\frac{n}{2}$) sequence.

Lemma 2.7. *Let $P(x) = \sum_{i=0}^{n} a_i x^i$, $Q(x) = \sum_{i=0}^{m} b_i x^i$ be symmetric, unimodal, and have non-negative coefficients. Then the same is true for their product $P(x)Q(x)$.*

Proof. Let $P_i(x) = x^i + x^{i+1} + \cdots + x^{n-i}$ and $Q_j(x) = x^j + x^{j+1} + \cdots + x^{m-j}$. Since $P(x)$ and $Q(x)$ are symmetric, we can write

$$P(x) = \sum_{i=0}^{[n/2]} (a_i - a_{i-1}) P_i(x) \text{ with } a_i - a_{i-1} \in \mathbb{N}$$

$$Q(x) = \sum_{j=0}^{[m/2]} (b_i - b_{i-1}) G_j(x) \text{ with } b_i - b_{i-1} \in \mathbb{N}$$

where $a_{-1} := 0$ and $b_{-1} := 0$.
Thus:

$$P(x)Q(x) = \sum_{i=0}^{[n/2]} \sum_{j=0}^{[m/2]} (a_i - a_{i-1})(b_i - b_{i-1}) P_i(x) Q_j(x).$$

It's obvious, that $P_i(x)Q_j(x)$ is unimodal with center $\frac{n+m}{2}$. Thus $P(x)Q(x)$ is a sum of symmetric unimodal polynomials with the same center, and therefore so it itself. This completes the proof. □

As an application, we conclude with:

Theorem 2.8. $\sum_{w \in \mathfrak{S}_n} x^{inv(w)}$ *is unimodal and symmetric.*

Here, $inv(w)$ is the *inversion number* of a permutation $w \in \mathfrak{S}_n$ defined as number of pairs $i < j$ such that $w(i) > w(j)$. We start the proof of this Theorem with the following well-known result:

Lemma 2.9. $\sum_{w \in \mathfrak{S}_n} x^{inv(w)} = (1 + x)(1 + x + x^2) \cdots (1 + x + x^2 + \cdots + x^{n-1})$.

Proof. Define the *inversion table* $I(w)$ of a permutation $w \in \mathfrak{S}_n$ as a vector $(a_1, a_2, \ldots, a_n)$ with $a_i = \#\{j > i : w(j) < w(i)\}$. Then the map $w \mapsto I(w)$ sets a bijection $\mathfrak{S}_n \leftrightarrow T_n = \{(a_1, a_2, \ldots, a_n) : 0 \le a_i \le n - i\}$ such that $inv(w) = a_1 + \cdots + a_n$. Hence,

$$\sum_{w \in \mathfrak{S}_n} x^{inv(w)} = \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} x^{a_1 + a_2 + \cdots + a_n} =$$

$$\left(\sum_{a_1=0}^{n-1} x^{a_1} \right) \left(\sum_{a_2=0}^{n-2} x^{a_2} \right) \cdots \left(\sum_{a_n=0}^{0} x^{a_n} \right) = (1 + x)(1 + x + x^2) \cdots (1 + x + x^2 + \cdots + x^{n-1}),$$

as desired. □

Proof of Theorem 2.8. Follows immediately from Lemma 2.9 and Lemma 2.7.

□

References

[1] Richard P. Stanley, *Enumerative Combinatorics*, Volume 1, 2nd edition.