HOME EXAM

IAP 2014: DIRECTED READING PROGRAM

Part I

1. Let G be the group of nonconstant linear transformations $x \mapsto ax + b$ over $\mathbb{F}_q$.
(a) Find all irreducible G-representations and compute their characters.
(b) Compute the tensor product of irreducible representations.

2. Let X_1, X_2 be two G-sets, so that $\mathbb{C}[X_1]$ and $\mathbb{C}[X_2]$ become G-representations.
(a) Find $c(X_1, X_2) := \dim \mathrm{Hom}_G(\mathbb{C}[X_1], \mathbb{C}[X_2])$.
(b) Let X_1 and X_2 be homogeneous G-spaces, so that $X_i = G/H_i$.
Show that $\mathbb{C}[X_i] = \mathrm{Ind}_{H_i}^G \mathbb{C}$ and prove that $c(X_1, X_2) = \# \ H_1 \backslash G/H_2$.
(c) Let us consider the action of $G \times G$ on G given by $(g_1, g_2) \circ g := g_1 g g_2^{-1}$, and let Reg be the corresponding $G \times G$-representation on $\mathbb{C}[G]$. Prove that $\dim \mathrm{Hom}_{G\times G}(\mathrm{Reg}, \mathrm{Reg})$ is equal to the number of G-conjugacy classes on the one hand, and to the number of irreducible finite dimensional G-representations on the other hand.

3. Let $G := \mathrm{GL}_n(\mathbb{F}_q)$ and $X_k^q := \mathrm{Gr}_k(\mathbb{F}_q^n)$–the space of k-dimensional subspaces of $\mathbb{F}_q^n$. The natural action of G on X_k^q induces a G-action on $\mathbb{C}[X_k^q] =: V_k^q$.
(a) Compute $\#G$ and $\#X_k^q$.
(b) Prove that G-representations V_k^q and V_{n-k}^q are isomorphic.
(c) Prove that $\dim \mathrm{Hom}_G(V_k^q, V_l^q) = 1 + \min\{k, l, n-k, n-l\}$.
(d) Prove that $V_k^q = \bigoplus_{i=0}^{\min\{k,n-k\}} U_i^q$ for some G-irreducible representations $U_0^q, \ldots, U_{[n/2]}^q$.

Part II

4. (a) Decompose $\mathrm{Ind}_{S_3}^{S_4}\pi$ into irreducibles for every irreducible S_3-representation π.
(b) Decompose $\mathrm{Ind}_{S_3\times S_2}^{S_5}\mathrm{sgn} \otimes \mathrm{sgn}$ into irreducibles.
(c) Decompose $\mathrm{Res}_{S_2\times S_2}^{S_4}\pi$ into irreducibles for every irreducible S_4-representation π.

5. Consider a subgroup $\mathbb{Z}_n \subset S_n$ generated by the long cycle $\sigma = (12\ldots n)$, and a character $\chi : \mathbb{Z}_n \to \mathbb{C}^*$ with $\chi(\sigma)$ being a primitive n-th root of 1.
(a) Decompose $\mathrm{Ind}_{\mathbb{Z}_n}^{S_n}\chi$ into irreducibles for $n = 3, 4$.
(b) Find the multiplicities of $V_{(1^n)} = \mathrm{sgn}$ and the standard representation $V_{(n-1,1)}$ in $\mathrm{Ind}_{\mathbb{Z}_n}^{S_n}\chi$.
(c) In general, show that the multiplicity of V_λ in $\mathrm{Ind}_{\mathbb{Z}_n}^{S_n}\chi$ is given by the following formula:

$$\frac{1}{n}\sum_{d|n}\mu(d)\chi_{V_\lambda}(\sigma^{n/d}),$$

where $\mu(d)$ is the Möbius function.

6. Let us define an element $C_n := \sum_{i<j} (ij) \in \mathbb{C}S_n$.

(a) Show that C_n acts on V_λ as a multiplication by the scalar $c_\lambda = \sum_j \sum_{i=1}^{\lambda_j} (i-j)$.
(b) Show that $E_n := (12)+\ldots+(1n) \in \mathbb{C}S_n$ acts diagonalizably on V_λ with integer eigenvalues from $\{1-n, 2-n, \ldots, n-2, n-1\}$.
(c) Show that E_n acts on V_λ as a multiplication by a scalar iff λ is a rectangular Young diagram. Compute this scalar in the latter case.

7. Recall that $\mathrm{Res}_{A_n}^{S_n} V_\lambda$ is irreducible iff $\lambda \neq \lambda^*$. If $\lambda = \lambda^*$ it decomposes into a sum of two conjugate A_n-irreducibles: $\mathrm{Res}_{A_n}^{S_n} V_\lambda = V'_\lambda \oplus V''_\lambda$. Compute the characters of V'_λ and V''_λ.

Hint: See [Fulton-Harris, Exercise 5.4] for the outline of key steps.

8. As we know, the group $G = \mathrm{SL}_2(\mathbb{F}_3)$ has 7 irreducible representations $\{V_i\}_{i=1}^7$. Let V_7 denote the representation $\mathrm{Res}_{\mathrm{SL}_2(\mathbb{F}_3)}^{\mathrm{GL}_2(\mathbb{F}_3)} X_\varphi$ in the notation of [Fulton-Harris, p. 70] (it is also recommended to check that this restriction is indeed irreducible and does not depend on φ).

Draw the graph, whose vertices are parametrized by $\{1, \ldots, 7\}$ and the number of edges between vertices $\#i$ and $\#j$ is equal to $\dim \mathrm{Hom}_G(V_j, V_i \otimes V_7)$.

9. Prove that $\mathrm{PSL}_2(\mathbb{F}_q)$ is a simple group for odd $q > 3$. Is $\mathrm{PSL}_2(\mathbb{F}_3)$ simple?

Part III

10. Classify irreducible finite dimensional representations of the two-dimensional Lie algebra with basis X, Y and commutation relation $[X, Y] = Y$. Consider the cases of zero and positive characteristic (we work only over algebraically closed fields).

11. Let L be a free Lie algebra on k generators $x_1, \ldots, x_k$. Consider a grading on L with $\deg(x_1) = \ldots = \deg(x_k) = 1$. Thus $L = \bigoplus_{n \geq 0} L_n$ with L_n being the degree n component. Prove the following formula:

$$\dim L_n = \frac{1}{n} \sum_{d|n} \mu(d) \cdot k^{n/d},$$

where $\mu(d)$ is the Möbius function.

Hint: Use the Möbius inversion formula.

12. Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$–its Cartan subalgebra. Let $R = R^- \cup R^+$ be the set of all roots of $\mathfrak{g}$ and $\Pi \subset R^+$ be the set of positive simple roots. Choose nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Pi$ and set $e := \sum_{\alpha \in \Pi} e_\alpha,\ h := \sum_{\alpha \in R^+} h_\alpha$. Show that there is a unique element $f \in \mathfrak{g}$ such that $\{f, h, e\}$ generate a subalgebra isomorphic to $\mathfrak{sl}_2$ via

$$F \mapsto f,\ H \mapsto h,\ E \mapsto e.$$