Lecture #6 09/10/2020

- Last time → Linear (in)dependence of a set S of vectors
  - → if $\vec{0}$ is in the set, then it this set is linearly dependent
  - → if S has more elements than their height ⇒ lin. dep.
  - → S - linearly indep. if the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution (A - matrix whose columns are all elements of S)

Ex 1: Let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} \in \mathbb{R}^4$ be a linearly independent set. Is it true that $\{\vec{v_3}, 2\vec{v_2}+\vec{v_1}, \vec{v_1}+\vec{v_3}\}$ is also lin. indep.

▸ Assume $\underbrace{x_1(\vec{v_3}) + x_2(2\vec{v_2}+\vec{v_1}) + x_3(\vec{v_1}+\vec{v_3})}_{(x_2+x_3)\vec{v_1} + 2x_2\cdot\vec{v_2} + (x_1+x_3)\cdot\vec{v_3}} = \vec{0}$

But $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ being lin. independent ⇒

$$\Rightarrow \begin{cases} x_2 + x_3 = 0 \\ 2x_2 = 0 \quad \Rightarrow x_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow x_3 = 0 \Rightarrow x_1 = 0$$

So: $x_1 = x_2 = x_3 = 0$.

Hence: $\{\vec{v_3}, 2\vec{v_2}+\vec{v_1}, \vec{v_1}+\vec{v_3}\}$ - linearly independent ∎

- → Transformations $T: \mathbb{R}^n \to \mathbb{R}^m$
  - → matrix transformation $T(\vec{x}) = A\vec{x}$

Ex 2: a) A is a 3x10 matrix. What are the values of a, b to define a map $T: \mathbb{R}^a \to \mathbb{R}^b$ by $T(\vec{x}) = A\vec{x}$?

b) How many rows AND columns a matrix A must have in order to define a map $\mathbb{R}^{11} \to \mathbb{R}^5$?

▸ a) $a = 10$, $b = 3$

b) 5 rows, 11 columns ∎

Ex 3: Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be defined via $T(\vec{x}) = \begin{pmatrix} -30 & 0 & 0 & 0 \\ 0 & -30 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -30 \end{pmatrix}\vec{x}$

Compute the image of $\vec{x} = \begin{pmatrix} 1/2 \\ 1/3 \\ -1/5 \\ -1/10 \end{pmatrix}$

▸ $T(\vec{x}) = A\vec{x} = \begin{pmatrix} -15 \\ -10 \\ 6 \\ 3 \end{pmatrix}$ ∎

Lecture # 6

Recall the definition from the very end of last time

Def: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$
2) $T(c\vec{u}) = c \cdot T(\vec{u})$ for any $\vec{u} \in \mathbb{R}^n$, $c \in \mathbb{R}$

Note that 1), 2) can be phrased as T preserving operations of vector addition and scalar multiplication.

Q: $T(\vec{0}) = ?$

(Pick any $\vec{v}$, then $\vec{0} = 0 \cdot \vec{v} \Rightarrow T(\vec{0}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \underline{\underline{\vec{0}}}$)

Q: $T(\vec{v_1} + \vec{v_2} + \vec{v_3}) = ?$ (assuming T - linear)

$\left(= T(\vec{v_1} + (\vec{v_2} + \vec{v_3})) = T(\vec{v_1}) + T(\vec{v_2} + \vec{v_3}) = \underline{T(\vec{v_1}) + T(\vec{v_2}) + T(\vec{v_3})} \right)$

Q: $T(c_1\vec{v_1} + c_2\vec{v_2} + \_\_\_ + c_k\vec{v_k}) = ?$ (assuming T - linear)

$\left(= T(c_1\vec{v_1}) + T(c_2\vec{v_2}) + \_\_\_ + T(c_k\vec{v_k}) = c_1T(\vec{v_1}) + c_2T(\vec{v_2}) + \ldots + c_kT(\vec{v_k}) \right)$.

Ex 5: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\5\\6\end{pmatrix}$

Compute $T\begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{pmatrix}1\\2\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\0\end{pmatrix} + 2 \cdot \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow T\begin{pmatrix}1\\2\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix} + 2T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix} + 2\begin{pmatrix}-2\\5\\6\end{pmatrix} = \begin{pmatrix}-3\\7\\14\end{pmatrix}$$

Q: Is the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (x_1 - 2x_2, 3x_3 + 2x_1 - 1, x_2 + x_3)$ linear?

(**A**: No, e.g. $T(\vec{0}) \neq \vec{0}$
Also note that both 1) & 2) fail!)

Lecture # 6

§1.9 The matrix of a linear transformation

Evoking our solution of EX5 on the previous page immediately gives rise to the following general result:

CLAIM: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Then, there exists a unique $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

Explicitly, the matrix A is given by

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{pmatrix}$$

← i.e. its columns are images of $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

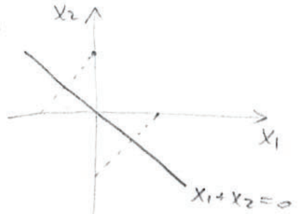
the standard matrix for the linear transformation T.

Reason: $T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T\left(x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right) = x_1 T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 T\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n T\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = A\vec{x}.$

Ex 6: a) Find the standard matrix for $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $\vec{x} \mapsto -5\vec{x}$

b) Find the standard matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ which reflects through the line $x_1 + x_2 = 0$

a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \Rightarrow A = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

b) 

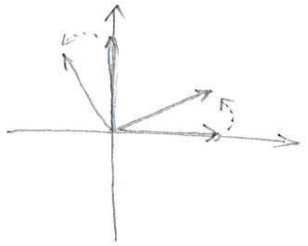
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

So, the general recipe is to evaluate images of the standard vectors $\vec{e}_1, \vec{e}_2, \dots$, and combine them into the matrix!

Lecture #6

The following very important example is treated in the textbook (pp 76-77) but let's still do it right now:

Ex 7: Find the standard matrix of $T: \mathbb{R}^2 \to \mathbb{R}^2$ – the rotation through the origin by angle φ counterclockwise.



$$\left.\begin{array}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \end{array}\right\} \Rightarrow \boxed{A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}$$

Q: Back to the question from Tuesday class, what is the geometric meaning of the matrix transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \vec{x} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

(A: rotation by $\pi/2$ counterclockwise)

! Look through examples on pages 78-80 in the textbook.

Ex 8: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined via $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ 2x_2 - 7x_3 \\ x_3 - 5x_1 \end{pmatrix}$.

Show that it is matrix transformation by finding standard matrix.

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & -7 \\ -5 & 0 & 1 \end{pmatrix}$$

Def: 1) A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to onto $\mathbb{R}^m$ if range$(T) = \mathbb{R}^m$, i.e. any $\vec{b} \in \mathbb{R}^m$ is an image of some $\vec{x} \in \mathbb{R}^n$.

2) A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$.

Rmk: If the reduced echelon form of A has "free variables", then the transformation $\vec{x} \mapsto A\vec{x}$ is NOT one-to-one.

Lecture #6

Let's conclude with the following two general results:

CLAIM: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.
Then, T is one-to-one iff $T(\vec{x}) = \vec{0}$ has only the trivial solution.

CLAIM: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transf., A - the standard matrix.

1) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.
2) T is one-to-one iff the columns of A are linearly independ.

(See pages 81-82 of the textbook for proofs - if interested)

Lecture #6

For the rest of today (to be continued next time), we shall discuss:

§2.1 Matrix Operations

First, given an $\underbrace{m\times n}_{m \text{ rows, } n \text{ columns}}$ matrix A, we use a_{ij} $\begin{pmatrix} 1\le i\le m \\ 1\le j\le n \end{pmatrix}$ to denote its entry in the i^{th} row & j^{th} column:

$$\boxed{A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}}$$

<u>Zero matrix 0</u>: if all $a_{ij}=0$

<u>Diagonal matrix</u>: if $m=n$ and $a_{ij}=0$ for $i\neq j$. i.e. $A=\begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$

Sum of matrices

If A, B are $m\times n$ matrices (they must be of the same size), then $\underbrace{A+B}_{\text{the sum}}$ is the $m\times n$ matrix, whose $(i,j)^{th}$ entry is $\underbrace{\text{the sum of } (i,j)^{th} \text{ entries of } A, B}_{a_{ij}+b_{ij}}$

Scalar Multiple of matrices

Given an $m\times n$ matrix A and a scalar $c\in\mathbb{R}$, the scalar multiple cA is the $m\times n$ matrix whose columns are c times columns of A, i.e. $(i,j)^{th}$ entry of cA equals $c\cdot a_{ij}$

<u>Ex 9</u>: Let $A=\begin{pmatrix} 1 & 2 \\ -3 & 5 \\ 7 & -4 \end{pmatrix}$, $B=\begin{pmatrix} -2 & 1 \\ 1 & 3 \\ -1 & 4 \end{pmatrix}$. Compute $2A-3B$

▶ $2A-3B=\begin{pmatrix} 8 & 1 \\ -9 & 1 \\ 17 & -20 \end{pmatrix}$ ▣

<u>Properties</u>:

A, B, C - of the same size

$c, d\in\mathbb{R}$ - scalars

1) $A+B=B+A$
2) $(A+B)+C=A+(B+C)$
3) $A+0=A=0+A$
4) $c\cdot(A+B)=c\cdot A+c\cdot B$
5) $(c+d)A=cA+dA$
6) $c(dA)=(cd)A$