Lecture #23

§6.1 Inner Product, Length, Angles, Orthogonality

We shall see now how familiar concepts from $\mathbb{R}^2$ (plane) or $\mathbb{R}^3$ (space) can be generalized to an arbitrary $\mathbb{R}^n$.

Def: Given two vectors $\vec{u}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$, $\vec{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$ in $\mathbb{R}^n$, the inner product (a.k.a. the dot product), denoted $\vec{u}\cdot\vec{v}$, is the number $\vec{u}^T\cdot\vec{v}$

Here: $\vec{u}^T$ is an $1\times n$ matrix, $\vec{v}$ is an $n\times 1$ matrix $\Rightarrow$ product $\vec{u}^T\cdot\vec{v}$ is a 1×1 matrix, i.e. a number.

Ex1: Compute the inner product of $\vec{u}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}5\\-3\\1\\-2\end{pmatrix}$ in $\mathbb{R}^4$.

▶ $\vec{u}\cdot\vec{v}=1\cdot 5+2\cdot(-3)+3\cdot 1+4\cdot(-2)=-6$ ▣

Properties of the inner product

1) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
2) $(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$
3) $(c\vec{u})\cdot\vec{v}=c(\vec{u}\cdot\vec{v})$
4) $\underbrace{\vec{u}\cdot\vec{u}}\geq 0$ (with equality iff $\vec{u}=0$)

for any $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$, $c\in\mathbb{R}$

if $\vec{u}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$, then $\vec{u}\cdot\vec{u}=u_1^2+u_2^2+\_\_+u_n^2$.

Def: Given a vector $\vec{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$ in $\mathbb{R}^n$, the length (a.k.a. the norm) of $\vec{v}$, denoted $\|\vec{v}\|$, is defined via $\|\vec{v}\|:=\sqrt{\vec{v}\cdot\vec{v}}=\sqrt{v_1^2+v_2^2+\_\_+v_n^2}\in\mathbb{R}_{\geq 0}$

Rmk: For $n=2,3$, this coincides with the usual notion of length, due to Pythagorean Thm.

Ex2: Find $\|\vec{u}\|$ and $\|\vec{v}\|$ in the setup of Ex1

▶ $\|\vec{u}\|=\sqrt{1^2+2^2+3^2+4^2}=\sqrt{30}$

$\|\vec{v}\|=\sqrt{5^2+(-3)^2+1^2+(-2)^2}=\sqrt{39}$ ▣

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Properties: 1) $\|c\cdot\vec{u}\| = \|\vec{u}\|\cdot \underbrace{|c|}_{\text{absolute value of } c}$

2) $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$ (the triangle inequality)

Def: A vector $\vec{v}$ is called a unit vector iff $\|\vec{v}\| = 1$.

! Given a nonzero vector $\vec{v}\in\mathbb{R}^n$, a unit vector in the direction of $\vec{v}$ is given by $\vec{u} := \frac{1}{\|\vec{v}\|}\cdot\vec{v}$, while a unit vector in the opposite direction is $\vec{w} = \frac{-1}{\|\vec{v}\|}\cdot\vec{v}$

Ex 3: Find a unit vector in the direction of $\vec{v} = \begin{pmatrix}3\\4\end{pmatrix}\in\mathbb{R}^2$

▶ $\vec{u} = \begin{pmatrix}3/5\\4/5\end{pmatrix}$ ∎

Def: Given a pair of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the distance b/w $\vec{u}$ and $\vec{v}$, denoted $\mathrm{dist}(\vec{u},\vec{v})$, is the length of $\vec{u}-\vec{v}$, i.e.

$$\mathrm{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

Ex 4: Compute $\mathrm{dist}(\vec{u},\vec{v})$ in the setup of Ex 1

▶ $\mathrm{dist}(\vec{u},\vec{v}) = \sqrt{(-4)^2+5^2+2^2+6^2} = \sqrt{81} = 9$ ∎

Def: Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$

Claim: For two vectors $\vec{u},\vec{v}$ in $\mathbb{R}^n$, we have
$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$, where θ – angle b/w $\vec{u}$ and $\vec{v}$

! In particular, two nonzero vectors in $\mathbb{R}^n$ are orthogonal iff the angle b/w them is $90° = \pi/2$

Note: $\vec{0}$ is orthogonal to any $\vec{u}\in\mathbb{R}^n$

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Claim: Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

[It's instructive to present a proof of this claim:

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \underbrace{\vec{u}\cdot\vec{u}}_{\|\vec{u}\|^2} + \underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2} + \underbrace{(\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u})}_{2\vec{u}\cdot\vec{v}} \Rightarrow \text{Claim follows}$$
]

The formula $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$ (θ - angle b/w $\vec{u}, \vec{v}$) is often used to determine the angle between two vectors

Ex 5: Find the angle θ b/w two vectors in Ex 1.

By Ex 1, Ex 2, have:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{-6}{\sqrt{30}\cdot\sqrt{39}} = -\frac{6}{3\cdot\sqrt{130}} = -\frac{2}{\sqrt{130}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-2/\sqrt{130}\right)$$

$\blacksquare$

Def: For a subspace W of $\mathbb{R}^n$, its orthogonal complement, denoted $W^\perp$, is the set of all $\vec{z} \in \mathbb{R}^n$ which are orthogonal to any $\vec{w} \in W$:

$$W^\perp = \{\vec{z} \in \mathbb{R}^n \mid \vec{z}\cdot\vec{w} = 0 \text{ for any } \vec{w} \in W\}$$

Q: If $W = \{\vec{0}\} \subseteq \mathbb{R}^n$, what is $W^\perp$?

A: $W^\perp = \mathbb{R}^n$

Q: If $W = \mathbb{R}^n$, what is $W^\perp$?

A: $W^\perp = \{\vec{0}\}$

Q: If $W \subseteq \mathbb{R}^2$ is a subspace consisting of all vectors lying on a given line ℓ through the origin, describe $W^\perp$.

A: $W^\perp$ is a subspace consisting of all vectors lying on an orthogonal line $\ell^\perp$ through the origin.

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Let A be an $m \times n$ matrix. Recall the following subspaces (discussed before):

$\mathrm{Col}\,A$ – subspace of $\mathbb{R}^m$

$\mathrm{Row}\,A$ – subspace of $\mathbb{R}^n$

$\mathrm{Nul}\,A$ – subspace of $\mathbb{R}^n$

Claim: $(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A$ and $(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T$

! Discuss a straightforward proof

Properties of $W^\perp$

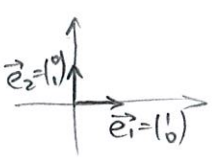
1) $W^\perp$ is a subspace
2) If W is spanned by $\vec{w}_1, \ldots, \vec{w}_k \in \mathbb{R}^n$, then $W^\perp$ consists of all $\vec{z} \in \mathbb{R}^n$ orthogonal to each $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$

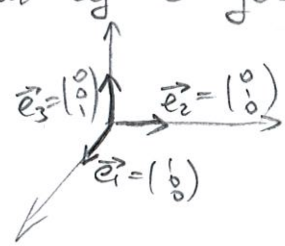
§6.2 Orthogonal Sets

Def: A set of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be an *orthogonal set* if any two distinct vectors in S are orthogonal:

$$\vec{u}_i \cdot \vec{u}_j = 0 \text{ for any } i \neq j$$

Examples:





Claim: If $S = \{\vec{u}_1, \ldots, \vec{u}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

Again, it is instructive to sketch the proof:

If $c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_k\vec{u}_k = \vec{0}$, then taking inner product with $\vec{u}_1$ we get $c_1 \cdot (\vec{u}_1 \cdot \vec{u}_1) + 0 + \ldots + 0 = 0 \Rightarrow c_1 = 0$

likewise: $c_2 = 0, \ldots, c_k = 0$

$\Rightarrow$ all weights $= 0$ $\Rightarrow S$ – lin. indep.

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Def: An *orthogonal basis* for a subspace $W \subseteq \mathbb{R}^n$ is a basis of W, which is also an orthogonal set.

Rmk: Due to the previous claim, orthogonal basis for W is an orthogonal set of elements of W, spanning W.

Claim: Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{y}$ in W, the weights in the linear combination

$$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_k\vec{u}_k$$

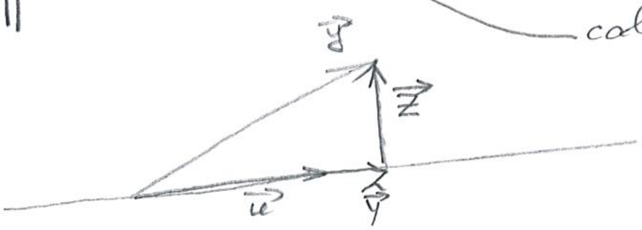
are given by

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} = \frac{\vec{y}\cdot\vec{u}_j}{\|\vec{u}_j\|^2}$$

Def: Given a nonzero vector $\vec{u}$ in $\mathbb{R}^n$, an *orthogonal projection* of a vector $\vec{y} \in \mathbb{R}^n$ onto $\vec{u}$ is the vector $\hat{y} \in \mathbb{R}^n$ such that

1) $\hat{y}$ is a multiple of $\vec{u}$
2) $\vec{z} := \vec{y} - \hat{y}$ is orthogonal to $\vec{u}$

called *component of $\vec{y}$ orthogonal to $\vec{u}$*



From 1), we see that $\hat{y} = \alpha\cdot\vec{u}$ for some constant α.

Taking inner product of $\vec{y} = \alpha\vec{u} + \vec{z}$ with $\vec{u}$, we get

$$\vec{y}\cdot\vec{u} = \alpha\cdot\vec{u}\cdot\vec{u} + 0 \Rightarrow \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}.$$

So: $\boxed{\hat{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\cdot\vec{u}}$

Note: The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the same as on any $c\cdot\vec{u}$, hence, the notation $\text{proj}_L\vec{y}$ for $\hat{y}$, where L - subspace spanned by $\vec{u}$.