Lecture #3

- Last time → Vectors, $\mathbb{R}^n$
  → Linear combinations, Span of a collection of vectors

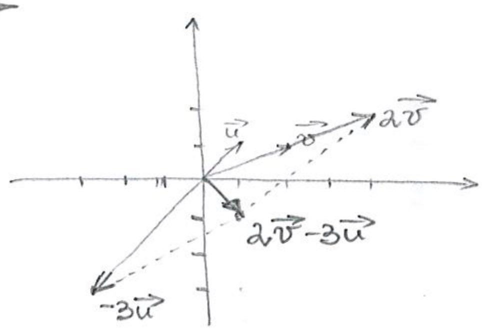
Recall: 1) Addition and scalar multiplication are done coordinate-wise
Warning: NOT TO CONFUSE with dot or cross products!

2) All vectors in this course are column vectors, i.e. we write them as a single column, which is NOT the same as a single row.
E.g. $\begin{bmatrix}1\\2\end{bmatrix} \neq [1\ 2]$

3) Given a vector $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$ we can identify it with a point in the plane

— // — $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ — // — a point in the space.

Ex1: Given $\vec{u} = \begin{pmatrix}1\\1\end{pmatrix}$, $\vec{v} = \begin{pmatrix}2\\1\end{pmatrix}$, visualize vectors $2\vec{v}$, $-3\vec{u}$, and $2\vec{v}-3\vec{u}$.



$$\left.\begin{aligned} 2\vec{v} &= \begin{pmatrix}4\\2\end{pmatrix} \\ -3\vec{u} &= \begin{pmatrix}-3\\-3\end{pmatrix}\end{aligned}\right\} \Rightarrow 2\vec{v}-3\vec{u} = \begin{pmatrix}1\\-1\end{pmatrix}$$

Note: All those vectors from above (i.e. $2\vec{v}, -3\vec{u}, 2\vec{v}-3\vec{u}$) are clearly linear combinations of $\vec{u}, \vec{v}$ (i.e. belong to Span$\{\vec{u},\vec{v}\}$).

Q: What about $\vec{0}$?
→ $\vec{0} \in$ Span$\{\vec{u},\vec{v}\}$ as $\vec{0} = 0\cdot\vec{u} + 0\cdot\vec{v}$ ▫

! We also did a bunch of True/False exprompt problems - see last page

! Warning: Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, the notation $\{\vec{v}_1, \ldots, \vec{v}_k\}$ just denotes this collection, while Span$\{\vec{v}_1, \ldots, \vec{v}_k\}$ denotes their span.
For example: $\begin{pmatrix}1\\1\end{pmatrix} \notin \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ BUT $\begin{pmatrix}1\\1\end{pmatrix} \in \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

Ex2: Given two nonzero vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, describe what Span$\{\vec{v}_1, \vec{v}_2\}$ looks like.
→ If $\vec{v}_1$ and $\vec{v}_2$ are proportional to each other, then Span$\{\vec{v}_1, \vec{v}_2\}$ is a line passing through the origin.
Otherwise, Span$\{\vec{v}_1, \vec{v}_2\}$ is a plane passing through the origin. ▫

The next exercise will allow us to relate vector equations

$$x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2} + \_\_\_ + x_k \cdot \vec{v_k} = \vec{w}$$

to linear systems discussed in Lecture 1.

Ex 3 : Let $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}$

Determine if $\vec{w}$ belongs to Span $\{\vec{v_1}, \vec{v_2}\}$.

another way to phrase: is $\vec{w}$ a linear combination of $\vec{v_1}, \vec{v_2}$.

▸ Using the definition of scalar multiplication and vector addition, this boils down to the question if there are scalars x_1, x_2 such that

$$x_1 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix} \text{ or equivalently } \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 8 \end{pmatrix}.$$

Hence, we need to verify if the linear system $\begin{cases} x_1 - 2x_2 = -4 \\ 3x_1 + x_2 = 9 \\ -2x_1 + 4x_2 = 8 \end{cases}$ is consistent or not.

Augmented matrix: $\left(\begin{array}{cc|c} 1 & -2 & -4 \\ 3 & 1 & 9 \\ -2 & 4 & 8 \end{array}\right) \xrightarrow[R_3 + 2R_1]{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 7 & 21 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{7}R_2} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$

$\downarrow R_1 \mapsto R_1 + 2R_2$

encodes linear system $\begin{cases} x_1 = 2 \\ x_2 = 3 \\ 0 = 0 \end{cases} \leftarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$

Thus, there is a unique solution: $x_1 = 2$, $x_2 = 3$.

Answer: Yes! (moreover, $\vec{w} = 2\vec{v_1} + 3\vec{v_2}$)

Note that we reduced the original question to the consistency of the linear system, whose augmented matrix is formed by placing column vectors $\vec{v_1}$ $\vec{v_2}$ $\vec{w}$ one after another.

CONCLUSION: A vector equation $x_1 \cdot \vec{a_1} + x_2 \cdot \vec{a_2} + \_\_\_ + x_k \cdot \vec{a_k} = \vec{b}$ has the same solution set as the linear system whose augmented matrix is $\left(\begin{array}{cccc|c} \vec{a_1} & \vec{a_2} & \dots & \vec{a_k} & \vec{b} \end{array}\right)$

In particular, $\vec{b}$ is a linear combination of $\vec{a_1}, \_ \vec{a_k}$ iff the corresponding linear system is consistent.

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Ex 4: Consider vectors $\vec{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ in $\mathbb{R}^3$.

For which values of parameter g, can $\vec{w} = \begin{pmatrix}2\\1\\g\end{pmatrix}$ be written as a linear combination of $\vec{v}_1, \vec{v}_2$.

▶ Boils down to consistency of the linear system with augmented matrix $\left(\begin{array}{cc|c}1 & 1 & 2\\2 & -1 & 1\\3 & 1 & g\end{array}\right)$. Performing elementary row operations, get:

$$\left(\begin{array}{cc|c}1 & 1 & 2\\2 & -1 & 1\\3 & 1 & g\end{array}\right) \underset{R_3 \mapsto R_3 - 3R_1}{\overset{R_2 \mapsto R_2 - 2R_1}{\rightsquigarrow}} \left(\begin{array}{cc|c}1 & 1 & 2\\0 & -3 & -3\\0 & -2 & g-6\end{array}\right) \overset{R_2 \mapsto \frac{1}{-3}R_2}{\rightsquigarrow} \left(\begin{array}{cc|c}1 & 1 & 2\\0 & 1 & 1\\0 & -2 & g-6\end{array}\right) \overset{R_3 \mapsto R_3 + 2R_2}{\rightsquigarrow}$$

$$\rightsquigarrow \left(\begin{array}{cc|c}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & g-4\end{array}\right) \overset{R_1 \mapsto R_1 - R_2}{\rightsquigarrow} \left(\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & g-4\end{array}\right) \Longrightarrow \text{corresponds to linear system} \begin{cases}x_1 = 1\\x_2 = 1\\0 = g-4\end{cases}$$

The latter is consistent iff $g=4$.

Answer: $g = 4$ ■

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§1.4 The matrix equation $A\vec{x} = \vec{b}$

We start from the following important definition which provides a convenient way to treat linear combinations of vectors

Def: If A is an $m \times n$ matrix, with n columns $\vec{a}_1, \dots, \vec{a}_n$ (of height m), then the product of A and $\vec{x} = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} \in \mathbb{R}^n$, denoted $A\vec{x}$, is the linear combination of the columns of A with weights being entries of $\vec{x}$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

Example: $\begin{pmatrix}1 & 3 & 10\\2 & -1 & 5\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\3\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix} + 3\begin{pmatrix}10\\5\end{pmatrix} = \begin{pmatrix}29\\20\end{pmatrix}$

$$\begin{pmatrix}1 & 2\\3 & -1\\10 & 5\end{pmatrix} \cdot \begin{pmatrix}1\\-2\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\3\\10\end{pmatrix} - 2 \cdot \begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}-3\\5\\0\end{pmatrix}$$

! If the height of $\vec{x}$ is not equal to the number of columns of A, then $A \cdot \vec{x}$ is NOT defined.

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In particular, vector equation $x_1\vec{a_1} + x_2\vec{a_2} + \dots + x_k\vec{a_k} = \vec{b}$ is equivalent to a matrix equation $\boxed{A\vec{x} = \vec{b}}$

$$A = \begin{pmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_k} \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$$

CONCLUSION: If A is an $m \times n$ matrix with columns $\vec{a_1}, \_\_, \vec{a_n}$, and $\vec{b} \in \mathbb{R}^m$, then the matrix equation $A\vec{x} = \vec{b}$ has the same solution set as the vector equation $x_1\vec{a_1} + \_\_ + x_n\vec{a_n} = \vec{b}$, which in turn has the same solution set as the linear system whose augmented matrix is $\left(\begin{array}{cccc|c} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} & \vec{b} \end{array}\right)$

Note: The equation $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is a linear combination of the columns of A.

Ex 5: Given three vectors $\vec{V_1}, \vec{V_2}, \vec{V_3} \in \mathbb{R}^4$, assume $2\vec{V_1} + 3\vec{V_2} + 7\vec{V_3} = 0$.
Find x_1, x_2 such that $\begin{pmatrix} \vec{V_1} & \vec{V_2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{V_3}$

$$2\vec{V_1} + 3\vec{V_2} + 7\vec{V_3} = 0 \Rightarrow 7\vec{V_3} = -2\vec{V_1} - 3\vec{V_2} \Rightarrow \vec{V_3} = -\frac{2}{7}\vec{V_1} - \frac{3}{7}\vec{V_2}.$$

Thus: can take $x_1 = -2/7$, $x_2 = -3/7$

In practice, the product $A\vec{x}$ is computed using the Row-Vector Rule:

Row-Vector Rule

If the product $A\vec{x}$ is defined (i.e. size of $\vec{x}$ equals number of columns of A) then the i^{th} entry of $A\vec{x}$ is the sum of products of the corresponding entries from the i^{th} row of A and from the vector $\vec{x}$

Examples:

$$\begin{pmatrix} 3 & 5 \\ -2 & 3 \\ 7 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\cdot 2 + 5\cdot 4 \\ (-2)\cdot 2 + 3\cdot 4 \\ 7\cdot 2 + 1\cdot 4 \end{pmatrix} = \begin{pmatrix} 26 \\ 8 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & 7 \\ 5 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + (-2)(-2) + 7\cdot 3 \\ 5\cdot 1 + 3\cdot(-2) + 1\cdot 3 \end{pmatrix} = \begin{pmatrix} 28 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

True / False : $\vec{0}$ is in the span of any collection $\vec{v}_1, \_\_ \vec{v}_k \in \mathbb{R}^n$

True / False : $\vec{w} \in \text{Span}\{\vec{v}_1, \_\_ \vec{v}_k\}$

$$\Downarrow$$

$$3\vec{w} \in \text{Span}\{\vec{v}_1, \_\_ \vec{v}_k\}$$

True / False : $\vec{w}_1, \vec{w}_2 \in \text{Span}\{\vec{v}_1, \_\_, \vec{v}_k\}$

$$\Downarrow$$

$$\vec{w}_1 + \vec{w}_2 \in \text{Span}\{\vec{v}_1, \_\_ \vec{v}_k\}$$

True / False : $\vec{w}_1 + \vec{w}_2 \in \text{Span}\{\vec{v}_1, \_\_ \vec{v}_k\}$

$$\Downarrow$$

$$\vec{w}_1, \vec{w}_2 \in \text{Span}\{\vec{v}_1, \_\_ \vec{v}_k\}$$

True / False : $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_3, \vec{v}_1, \vec{v}_2\}$