

Lecture #1

MA 42500 course "Elements of Complex Analysis"

In school: $\mathbb{N} = \{1, 2, 3, \dots\}$ $\downarrow$ to subtract bigger from smaller

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

 $\downarrow$ to divide numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

 $\downarrow$

$$\mathbb{R} = \{\text{all real numbers}\}$$

 $\downarrow$

$$\mathbb{C} \leftarrow \text{main object for this course, which we shall introduce now.}$$

Let i be the formal element, which is a solution of equation $x^2 = -1$.

Def. A complex number is an expression of the form $z = a + bi$, where a, b - real numbers. Moreover, a is called the real part of $z = a + bi$ while b is called the imaginary part of z .

Notations: $\mathbb{C} = \{\text{the set of all complex numbers}\}$

$$a = \operatorname{Re}(z), b = \operatorname{Im}(z) \text{ for } z = a + bi$$

Recall: On $\mathbb{Q}$ and $\mathbb{R}$ we had two basic operations "+", "•" subject to a bunch of relations, such as

$$a + b = b + a, \quad a + (b + c) = (a + b) + c$$

$$a \cdot b = b \cdot a, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

as well as the element $1, 0$ s.t.

$$a \cdot 1 = a = 1 \cdot a, \quad a + 0 = a = 0 + a$$

and any element $a \neq 0$ had inverse a^{-1} such that

$$a \cdot a^{-1} = 1 = a^{-1} \cdot a$$

Same is true on $\mathbb{C}$ as we shall see now.

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Addition: $\left. \begin{array}{l} z_1 = a_1 + b_1 i \\ z_2 = a_2 + b_2 i \end{array} \right\} \Rightarrow \text{set } z_1 + z_2 = \boxed{(a_1 + a_2) + (b_1 + b_2)i}$

Multiplication: $z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) \xrightarrow[\text{I use } i^2 = -1]{\text{open brackets}} \boxed{(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i}$

Have exercise: verify the first 5 properties from p.1.

Q: What about division $\frac{z}{w}$?

Let's illustrate this first by computing z^{-1} for $z \neq 0$.

If $z = x + yi \Rightarrow \frac{1}{z} = \frac{1}{x + yi} = \frac{1}{x + yi} \cdot \frac{x - yi}{x - yi} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i$

So: $\boxed{(x + yi)^{-1} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i}$

Ex: Write $\frac{-1 + 4i}{2 + i}$ in the form $a + bi$

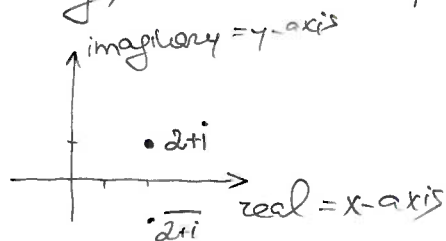
$\Rightarrow \frac{-1 + 4i}{2 + i} = \frac{-1 + 4i}{2 + i} \cdot \frac{2 - i}{2 - i} = \frac{2 + 9i}{5} = \frac{2}{5} + \frac{9}{5}i$

Ex: Find i^{2025} .

$\Rightarrow i^2 = -1 \Rightarrow i^4 = 1 \Rightarrow i^{2024} = \underbrace{i^4 \cdot \dots \cdot i^4}_{506} = 1 \Rightarrow i^{2025} = i^{2024} \cdot i = i$

Q: How one can visualize $\mathbb{C}$?

Clearly, we can depict a complex number by a point in a plane



Note: Addition of complex numbers is then the same as usual addition of vectors!

Def: A complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$

We already utilized this in division as $z \cdot \bar{z} = (a + bi)(a - bi) = \underbrace{a^2 + b^2}_{\text{real!}}$

Geometrically, $\bar{z}$ is obtained from z by reflection in real axis.

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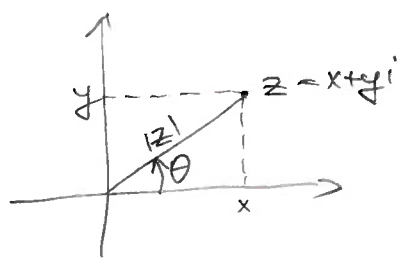
Q: What is the geometric interpretation of product?

Somehow, the formula

$$(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$$

doesn't seem to have a simple meaning in usual coordinates.

Hint: Switch to polar coordinates.



$$z = |z| \cdot \cos \theta + i \cdot |z| \sin \theta = |z| \cdot (\cos \theta + i \sin \theta)$$

Def: $|z| = \sqrt{x^2 + y^2} = \text{absolute value or modulus of } z$

As per the angle θ , note that adding 2π we actually get the same point on the plane.

Def: a) The principal value of the argument of $z \in \mathbb{C}$ is θ as in picture satisfying $-\pi < \theta \leq \pi$. This is denoted $\text{Arg}(z)$.
b) The argument of z , denoted $\arg(z)$, is any angle of the form $\text{Arg}(z) + 2\pi k$ with $k \in \mathbb{Z}$.

Examples: $\text{Arg}(-1) = \pi$, $\text{Arg}(-1-i) = -\frac{3\pi}{4}$.

Note: $z \cdot \bar{z} = |z|^2$

Product revised

If $z_1 = |z_1| \cdot (\cos \theta_1 + i \sin \theta_1)$, $z_2 = |z_2| \cdot (\cos \theta_2 + i \sin \theta_2)$, then

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \left(\underbrace{(\cos \theta_1 \cdot \cos \theta_2 - \sin \theta_1 \cdot \sin \theta_2)}_{=\cos(\theta_1 + \theta_2)} + \underbrace{(\cos \theta_1 \cdot \sin \theta_2 + \cos \theta_2 \cdot \sin \theta_1)}_{=\sin(\theta_1 + \theta_2)} i \right)$$

$$= |z_1| |z_2| \cdot (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

So: Under the product, absolute values multiply, arguments add.

Warning: $\text{Arg}(z_1 \cdot z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$ as right side may be $> \pi$.

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In this course we shall be mostly doing calculus on functions of a complex variable!

Let's conclude by a basic result (to be proved later in class):

Theorem: Any polynomial of degree n (in a variable z) has exactly n roots, if counted with multiplicities

Remarks: 1) $\text{Arg } z$ is not continuous near $\mathbb{R}_{<0}$!

2) Generalizing the notion $\text{Arg } z$, we could define $\forall \tau \in \mathbb{R}$:
 $\arg_{\tau} z$ to be the unique value in $\arg z$ s.t.

$$\tau < \arg_{\tau} z \leq \tau + 2\pi$$

(in particular, $\text{Arg } z = \arg_{-\pi} z$)

Last in-class checks (both algebraically and geometrically)

- verify $\overline{\overline{z}} = z$
- verify $\overline{z+w} = \overline{z} + \overline{w}$
- verify $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$