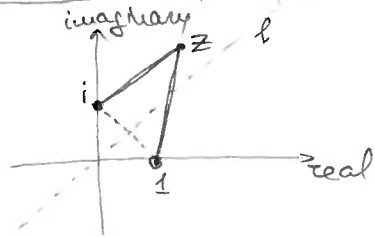


Lecture #2

Last time: Complex numbers, complex plane, polar coordinates.

Ex1: Describe all complex z such that $|z-i| = |z-1|$.

Geometric proof



$|z-i|$ = distance b/w z and i

$|z-1|$ = distance b/w z and 1

So: We are looking for points on a plane s.t. distances from them to i and 1 are the same. and those are easily seen to lie on the line above. Hence $z = \{x + x \cdot i \mid x \text{ real}\}$.

Algebraic Proof

$$\text{Write } z = x + yi \Rightarrow \begin{cases} z-i = x + (y-1)i \\ z-1 = (x-1) + yi \end{cases} \Rightarrow \sqrt{x^2 + (y-1)^2} = \sqrt{(x-1)^2 + y^2}$$

$$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 - 2x + 1 + y^2 \Rightarrow x = y. \text{ So } z = x + x \cdot i \text{ with any } x \in \mathbb{R}$$

Last time to each $z = x + yi \in \mathbb{C}$ we associated $\text{Re}(z)$, $\text{Im}(z)$, $\bar{z}$.

Note: $\text{Re } z = \frac{z + \bar{z}}{2}$
 $\text{Im } z = \frac{z - \bar{z}}{2i}$
 $\overline{\bar{z}} = z$

Q1: Which of the following are true:

a) $\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

b) $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

Q2: Given a complex $z \neq 0$, describe $\text{Arg}(\bar{z})$, $\text{Arg}(z^{-1})$ via $\text{Arg}(z)$.
 Similarly, describe $\arg(\bar{z})$, $\arg(z^{-1})$ via $\arg(z)$.

Warning: $\arg(z)$ is a multi-valued function, while $\text{Arg}(z)$ is not continuous

↑ we will get to this next time

Lecture #2Today: § 1.4-1.5

Question: Can one extend the "exponent" to a complex-valued function on a complex plane $e^z: \mathbb{C} \rightarrow \mathbb{C}$?

Basic Properties of Exponent:

$$e^z \cdot e^w = e^{z+w}, \quad e^0 = 1, \quad (e^z)' = e^z. \quad \leftarrow \text{want to keep all these properties}$$

Let's now take $z = x + y \cdot i$ with $x, y \in \mathbb{R}$. Then first we have $e^z = e^x \cdot e^{iy}$.

Invoking the Taylor series expansion, we get:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots = 1 + iy - \frac{y^2}{2!} - i \cdot \frac{y^3}{3!} + \frac{y^4}{4!} - \dots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)i = \cos y + i \cdot \sin y \end{aligned}$$

which results in Euler's equation:

$$e^{i\theta} = \cos \theta + i \cdot \sin \theta \quad \text{for any } \theta \in \mathbb{R}$$

In particular, for $\theta = \pi$, we get the famous formula $e^{i\pi} = -1$

Def: We define a complex exponent as follows: $e^{x+yi} = e^x (\cos y + i \sin y)$

At the moment, you can just think of $e^{i\theta}$ as a "shortcut" notation for $\cos \theta + i \cdot \sin \theta$. In particular, for $z \neq 0$, we get

$$z = |z| \cdot e^{i \cdot \arg(z)}, \quad \text{where } |z|, \arg(z) \text{ - polar coordinates}$$

Hence, we can summarize the end of previous lecture as follows:

$$\text{if } z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2} \Rightarrow \begin{cases} z_1 \cdot z_2 = r_1 r_2 \cdot e^{i(\theta_1 + \theta_2)} \\ z_1 / z_2 = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)} \end{cases}$$

Also, evoking $\operatorname{Re}(e^{i\theta}) = \cos \theta$, $\operatorname{Im}(e^{i\theta}) = \sin \theta$, we thus get

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Lecture #2

Let us now prove the famous De Moivre's formula:

$$\boxed{(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)} \quad \text{for any } n \in \{1, 2, 3, \dots\}$$

$\Rightarrow \cos \theta + i \sin \theta = e^{i\theta}$, and $(e^{i\theta})^n = \underbrace{e^{i\theta} \cdot e^{i\theta} \cdot \dots \cdot e^{i\theta}}_{n \text{ times}} = e^{i \cdot n\theta} = \cos(n\theta) + i \sin(n\theta)$

This allows to express $\cos(n\theta)$, $\sin(n\theta)$ via $\cos \theta$, $\sin \theta$.

Ex2: Evaluate $\int_0^\pi \sin^4 \theta d\theta$.

$\triangleright$ This is actually a school-level problem, based on formula for $\sin^2 \theta$.

But for those who forgot the formula, let's derive it.

$$\begin{aligned} \sin^4 \theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^4 = \frac{1}{16} (e^{i\theta} - e^{-i\theta})^4 = \frac{e^{i \cdot 4\theta} - 4 \cdot e^{i \cdot 2\theta} + 6 - 4 \cdot e^{i(-2\theta)} + e^{i(-4\theta)}}{16} \\ &= \frac{2\cos(4\theta) - 8\cos(2\theta) + 6}{16} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^\pi \sin^4 \theta d\theta &= \int_0^\pi \frac{1}{16} (2\cos(4\theta) - 8\cos(2\theta) + 6) d\theta = \frac{1}{16} \left[\frac{1}{2} \sin(4\theta) - 4\sin(2\theta) + 6\theta \right]_0^\pi \\ &= \frac{3\pi}{8} \end{aligned}$$

Ex3: Evaluate $(1-i)^{2025}$

$\triangleright 1-i = \sqrt{2} \cdot e^{i(-\frac{\pi}{4})} \Rightarrow (1-i)^{2025} = \sqrt{2}^{2025} \cdot e^{i \cdot (-\frac{\pi}{4}) \cdot 2025}$

But $-\frac{2025\pi}{4} = -506\pi - \frac{\pi}{4} = 2\pi \cdot (-253) - \frac{\pi}{4}$

$\Rightarrow e^{i \cdot (-\frac{\pi}{4}) \cdot 2025} = \cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

So, $(1-i)^{2025} = \sqrt{2}^{2025} \cdot \frac{1-i}{\sqrt{2}} = \boxed{2^{1012} - i \cdot 2^{1012}}$

Note: If $z = re^{i\theta}$, then $z^{-1} = r^{-1} \cdot e^{i(-\theta)} \Rightarrow z^{-n} = (z^{-1})^n = r^{-n} \cdot e^{i(-n\theta)}$ for $n \in \{1, 2, 3, \dots\}$
so that formula $z^n = r^n \cdot \underbrace{(\cos(n\theta) + i \sin(n\theta))}_{e^{i \cdot n\theta}}$ holds for all $n \in \mathbb{Z}$.

Lecture #2

§1.5 (start)

Let's now learn how to extract n^{th} roots of complex numbers.

Q: Solve $z^2 = 1$

$$z = \pm 1$$

$$z^4 = 1$$

$$z = \pm 1, \pm i$$

$$z^n = 1$$

let's write z in polar coordinates $z = re^{i\theta}$
 $\Rightarrow z^n = r^n \cdot e^{in\theta} \Rightarrow z^n = 1$ iff $\begin{cases} r^n = 1 \\ n\theta = 2\pi k, k\text{-integer} \end{cases}$

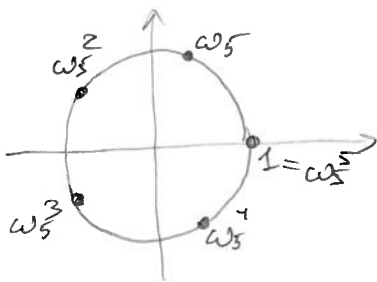
r - non-negative real, $r^n = 1 \Rightarrow r = 1$

$$n\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{n}$$

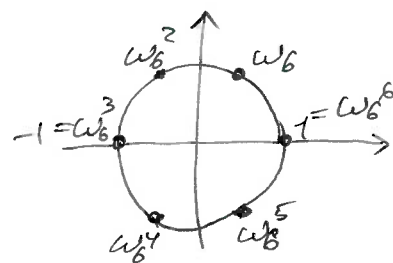
BUT increasing k by n changes $\theta \rightarrow \theta + 2\pi$
 which corresponds to the same complex number

Thus: we get n roots of $z^n = 1$ which correspond to vertices of regular n -gon inscribed in the unit circle.

E.g. $z^5 = 1$ has roots $1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$



with $\boxed{\omega_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)}$



E.g. $z^6 = 1$ has roots $1, \omega_6, \omega_6^2, \omega_6^3, \omega_6^4, \omega_6^5$

Ex 4: Solve $z^4 = 3i$

$$z = re^{i\theta} \Rightarrow z^4 = r^4 \cdot e^{i4\theta} \Rightarrow \begin{cases} r^4 = 3 \\ 4\theta = \frac{\pi}{2} + 2\pi k, k\text{-integer} \end{cases} \Rightarrow \begin{cases} r = \sqrt[4]{3} \\ \theta = \frac{\pi}{8} + \frac{\pi k}{2}, k\text{-integer} \end{cases}$$

But: changing $k \rightarrow k+4$ changes $\theta \rightarrow \theta + 2\pi$ hence the same z .

So: $z = \sqrt[4]{3} \left(\cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + i \sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) \right)$ with $k = -2, -1, 0, 1$
 you could take any other 4 consecutive integers, but above choice is just to get in $[-\pi, \pi]$