

Lecture #3

We shall start today's lecture with practicing on Examples first.

Ex1: Show that $-2+i$ is a solution of $z^2+4z+5=0$.

$$\begin{aligned} (-2+i)^2 + 4(-2+i) + 5 &= 4 - 4i + \underbrace{i^2}_{=-1} - 8 + 4i + 5 = 0 \end{aligned}$$

[Note: We can actually find all solutions in the usual way:

$$0 = z^2 + 4z + 5 = (z+2)^2 + 1 \Rightarrow (z+2)^2 = -1 \Rightarrow z+2 = \pm i \Rightarrow z = -2 \pm i.$$

Ex2: Find the principal argument and write in polar coordinates $\frac{1-i}{1+\sqrt{3}i}$

$$\begin{aligned} \left. \begin{aligned} 1-i &= \sqrt{2} \cdot e^{i(-\pi/4)} \\ 1+\sqrt{3}i &= 2 \cdot e^{i\pi/3} \end{aligned} \right\} \Rightarrow \frac{1-i}{1+\sqrt{3}i} &= \frac{1}{\sqrt{2}} e^{i(-\pi/4 - \pi/3)} = \boxed{\frac{1}{\sqrt{2}} e^{i(-7\pi/12)}} \Rightarrow \text{Arg} \left(\frac{1-i}{1+\sqrt{3}i} \right) = -\frac{7\pi}{12} \\ &\text{as it is in } (-\pi; \pi] \end{aligned}$$

[Note: If you tried to write this as $a+bi$, you wouldn't see such answer:

$$\frac{1-i}{1+\sqrt{3}i} = \frac{1-i}{1+\sqrt{3}i} \cdot \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{1-\sqrt{3} - (1+\sqrt{3})i}{4}$$

Ex3: Write $-4\pi(1+\sqrt{3}i)$ in polar form $re^{i\theta}$.

$$\begin{aligned} r = |z| &= \sqrt{(-4\pi)^2 + (-4\pi\sqrt{3})^2} = 8\pi \\ \cos \theta + i \sin \theta = \frac{z}{r} &= \frac{-4\pi(1+\sqrt{3}i)}{8\pi} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \Rightarrow \theta = \frac{4\pi}{3} \end{aligned} \quad \left\{ \boxed{8\pi e^{i\frac{4\pi}{3}}} \right.$$

[Note: If the question asked for the principal argument θ , then $\frac{4\pi}{3}$ wouldn't work as it is $> \pi$, and we would write $8\pi \cdot e^{i(-\frac{2\pi}{3})}$

Ex4: Express $\cos(4\theta)$ via $\cos \theta, \sin \theta$.

$$\cos(4\theta) = \text{Re}((\cos \theta + i \sin \theta)^4) \text{ by De Moivre's Formula. } \left\{ \Rightarrow \right.$$

$$\text{Recall: } (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\Rightarrow \cos(4\theta) = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta.$$

[Note: One could further simplify using $\cos^2 \theta + \sin^2 \theta = 1$.

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Ex5: Let $n \geq 2$ be integer and $\omega_n = e^{i \frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$. Prove that $1 + \omega_n + \omega_n^2 + \dots + \omega_n^{n-1} = 0$.

Proof #1: By geometric progression formula:

$$1 + \omega_n + \dots + \omega_n^{n-1} = \frac{1 - \omega_n^n}{1 - \omega_n} = \frac{1 - e^{i \cdot \frac{2\pi}{n} \cdot n}}{1 - \omega_n} = 0 \quad \checkmark$$

Proof #2 (equivalent to above): $(1 - \omega_n)(1 + \omega_n + \omega_n^2 + \dots + \omega_n^{n-1}) =$
 $1 + \omega_n + \omega_n^2 + \dots + \omega_n^{n-1} - \omega_n - \omega_n^2 - \dots - \omega_n^{n-1} - \omega_n^n = 1 - \omega_n^n = 0 \quad \checkmark$

Geometric/Physics proof #3: The points $1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}$ are vertices of a regular n -gon encircled by the unit circle. Hence, the sum is invariant under rotation by $\uparrow \frac{2\pi}{n}$, hence is zero (i.e. center of mass is 0 due to the symmetry!)

Ex6: For any integer $n \geq 2$, evaluate $1 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2\pi(n-1)}{n}$.

$\cos \frac{2\pi}{n} = \operatorname{Re} \omega_n$ and likewise $\cos \frac{2\pi \cdot k}{n} = \operatorname{Re}(\omega_n^k) \Rightarrow$

$$\Rightarrow 1 + \cos \frac{2\pi}{n} + \dots + \cos \frac{2\pi(n-1)}{n} = \operatorname{Re}(1 + \omega_n + \omega_n^2 + \dots + \omega_n^{n-1}) \stackrel{\text{Ex5}}{=} \operatorname{Re}(0) = 0$$

!) Note that quadratic equations can be solved the same way as in school.

Namely, if $a, b, c \in \mathbb{C}$ and $a \neq 0$, then $az^2 + bz + c = 0$ can be written as

$a(z + \frac{b}{2a})^2 + c - \frac{b^2}{4a} = 0$, or as $(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$. Thus, solutions are:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $b^2 - 4ac < 0$, there is no preferred choice of two possible values of $\sqrt{b^2 - 4ac}$, and so one can really write above as

$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ where $\sqrt{\dots}$ is understood as all square roots!

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Today: §1.6.

Recall that in usual calculus, one studies $f: \mathbb{R} \rightarrow \mathbb{R}$. And for limits, continuity, one usually appeals to " ε -neighborhoods" of a point:
 $(x_0 - \varepsilon, x_0 + \varepsilon)$ with $\varepsilon > 0$



Now, when studying functions $f: \mathbb{C} \rightarrow \mathbb{C}$ of a complex variable it's quite natural to replace the above with

$$D_\varepsilon(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\} = \text{open disk of radius } \varepsilon > 0 \text{ centered at } z_0$$



Note: $|z - z_0|$ = distance between z_0 & z .

Def: a) A point z_0 of a set S in $\mathbb{C}$ is an interior point if there is some $\varepsilon > 0$ such that $D_\varepsilon(z_0)$ is in S , denoted $D_\varepsilon(z_0) \subseteq S$.

b) A set S in the complex plane $\mathbb{C}$ is open if each point is interior.

Examples:  - open,  - open,  - not open

Note: the value of ε in b) does depend on the point z_0 !

Def: An open set S in $\mathbb{C}$ is (path) connected if any two points of S can be joined by a continuous curve in this set S .



Note: In fact, one can always make this curve to be piece-wise linear (= polygonal path)

Def: A set Ω in $\mathbb{C}$ is a domain if it is open AND (path) connected.

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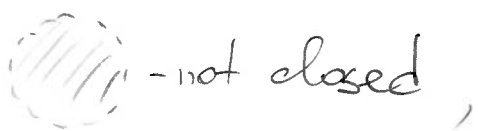
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True / False:

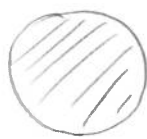
- a) If U_1, U_2 - open sets of $\mathbb{C}$, then so is their union $U_1 \cup U_2$
- b) If U_1, U_2 - open sets of $\mathbb{C}$, then so is their intersection $U_1 \cap U_2$
- c) If $\{U_n\}_{n=1}^{\infty}$ - open sets of $\mathbb{C}$, then so is $\bigcup_{n=1}^{\infty} U_n$.
- d) If $\{U_n\}_{n=1}^{\infty}$ - open sets of $\mathbb{C}$, then so is $\bigcap_{n=1}^{\infty} U_n$.

Def: a) A point $z_0 \in \mathbb{C}$ is called a boundary point of a set $S \subset \mathbb{C}$ if for any $\varepsilon > 0$ the open disk $D_\varepsilon(z_0)$ contains a point from S as well as a point not in S .

b) A set S of $\mathbb{C}$ is closed if it contains all of its boundary points.



- not closed,



- closed,



- closed

True / False: S - closed if and only if its complement $\mathbb{C} \setminus S$ is open.