

Lecture #5

Ex 1: Verify that $w = f(z) = \frac{1}{z}$ maps:

a) circle $|z|=3$ to a circle $|w|=\frac{1}{3}$

b) circle $|z-1|=1$ to the vertical line $x=\frac{1}{2}$

a) If $|z|=3$, then $|w|=\frac{1}{3}$ clearly. Backwards, for any $|w|=\frac{1}{3}$, if we set $z=w^{-1}$, then $|z|=3$ and $w=f(z)$.

Alternative way: $z=3e^{i\theta} \Rightarrow w=\frac{1}{3}e^{i(-\theta)} \Rightarrow$ has range as claimed.

b) parametrize $z=1+\cos\theta+i\sin\theta$, $-\pi < \theta \leq \pi$. Then:

$$w = \frac{1}{1+\cos\theta+i\sin\theta} = \frac{1+\cos\theta-i\sin\theta}{(1+\cos\theta)^2+(\sin\theta)^2} = \frac{(1+\cos\theta)-i\sin\theta}{2(1+\cos\theta)} = \frac{1}{2} - i \frac{\sin\theta}{2(1+\cos\theta)}$$

So the image is indeed on a vertical line $x=\frac{1}{2}$.

To finish the proof we need to show $\frac{\sin\theta}{1+\cos\theta}$ can take any real value.

By l'Hopital rule: $\lim_{\theta \rightarrow \pi^-} \frac{\sin\theta}{1+\cos\theta} = +\infty$, $\lim_{\theta \rightarrow -\pi^+} \frac{\sin\theta}{1+\cos\theta} = -\infty$, while

the function $\frac{\sin\theta}{1+\cos\theta}$ is continuous on $(-\pi, \pi)$, hence, it achieves each value. (Alternatively: $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, $1+\cos\theta = 2\cos^2\frac{\theta}{2} \Rightarrow \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$ - takes all values!)

Note: Unlike for functions $f: \mathbb{R} \rightarrow \mathbb{R}$ in school where you could be often asked to draw a graph, this is not quite doable in the present setup of $f: \mathbb{C} \rightarrow \mathbb{C}$. However, we can still ask you to draw domain and range of f .

Ex 2: Let $z \in \mathbb{C}$ satisfy $|z| < 1$ and set $z_n = 1+z+\dots+z^{n-1}$. Find $\lim_{n \rightarrow \infty} z_n$.

By geometric progression formula $z_n = \frac{1-z^n}{1-z}$

(if you forgot: $(1-z)(1+z+\dots+z^{n-1}) = 1+z+z^2+\dots+z^{n-1} - z - z^2 - \dots - z^{n-1} - z^n = 1-z^n$)

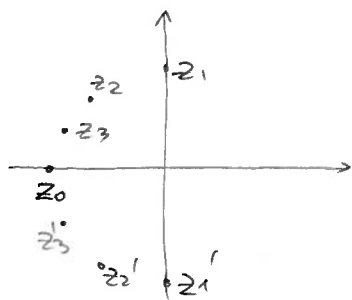
We claim that $\lim_{n \rightarrow \infty} \frac{1-z^n}{1-z} = \frac{1}{1-z}$. Indeed, we have

$$\left| \frac{1-z^n}{1-z} - \frac{1}{1-z} \right| = \frac{|z|^n}{|1-z|} \xrightarrow{n \rightarrow \infty} 0 \text{ as } |z| < 1$$

Lecture #5

Note: If the limit $\lim_{z \rightarrow z_0} f(z)$ exists, then for any sequence of points $\{z_n\}_{n=1}^{\infty} \rightarrow z_0$ we should have the same value of $\lim_{n \rightarrow \infty} f(z_n)$.

Ex 3: Show that $\text{Arg } z$ is discontinuous at $\mathbb{R}_{<0}$.



If we take $z_n = |z_0| e^{i(\pi - \frac{\pi}{n})}$, then $z_n \rightarrow z_0$ and $\text{Arg}(z_n) \rightarrow \pi$, while if we take $z'_n = |z_0| e^{-i(\pi - \frac{\pi}{n})}$, then $z'_n \rightarrow z_0$ and $\text{Arg}(z'_n) \rightarrow -\pi$. As $\pi \neq -\pi$, there is no $\lim_{z \rightarrow z_0} \text{Arg } z$.

Let's now recall the very last example from last time:

Example: $\lim_{z \rightarrow i} \frac{z^2 + 1}{z(z-i)}$

Note that both numerator and denominator vanish at $z=i$, so we cannot just plug in. However, $z^2 + 1 = (z-i)(z+i)$, hence $\frac{z^2 + 1}{z(z-i)} = \frac{z+i}{z}$. Hence $\lim_{z \rightarrow i} f(z) = \frac{2i}{i} = 2$. Note though that $f(z)$ is not defined at $z=i$!

Def: A function $f(z)$ has a removable discontinuity at z_0 if it becomes continuous at z_0 after redefining value $f(z_0)$.

Note that in contrast to $f: \mathbb{R} \rightarrow \mathbb{R}$ where $\lim_{x \rightarrow x_0} f(x)$ is the common value of right limit $\lim_{x \rightarrow x_0^+} f(x)$ and left limit $\lim_{x \rightarrow x_0^-} f(x)$, in the present setup we actually "approach $z_0 \in \mathbb{C}$ from any direction"



Q: $\lim_{z \rightarrow i} \frac{z^2 + 3z}{z^2 + 1} = ?$, $\lim_{z \rightarrow \infty} \frac{2z+2}{iz+4} = ?$, $\lim_{z \rightarrow \infty} \frac{2z^2+2}{iz+4} = ?$

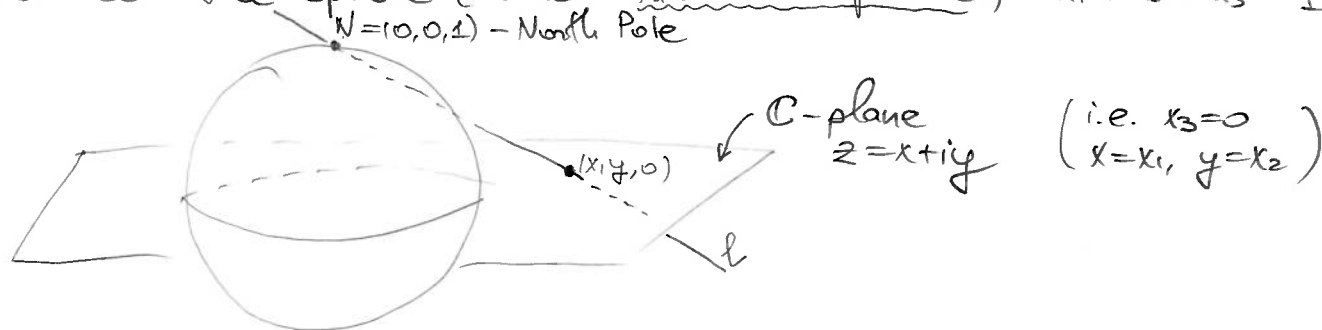
To answer this question we first need to discuss what " ∞ " means.

Lecture #5

13

§1.7: Riemann sphere

Consider the sphere (called Riemann sphere) $x_1^2 + x_2^2 + x_3^2 = 1$



We have a natural bijection between points on the plane $x_3=0$ (which we think of as a complex plane) and points on the sphere excluding N , obtained by intersecting any line passing through N and a point in $\mathbb{C}$ -plane with a sphere. Let's derive explicit formulas.

Lemma 1: A point $z = x+iy \in \mathbb{C} = \{x_3=0 \text{ plane}\}$ corresponds to the point (x_1, x_2, x_3) on above sphere with $x_1 = \frac{2 \operatorname{Re} z}{|z|^2 + 1}$, $x_2 = \frac{2 \operatorname{Im} z}{|z|^2 + 1}$, $x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$.

Any point on the line l passing through $N=(0,0,1)$ and $(x,y,0)$ can be parametrized by $(x_1, x_2, x_3) = (0,0,1) + t \cdot (x,y,-1)$ with $t \in \mathbb{R}$, i.e.
 $x_1 = tx$, $x_2 = ty$, $x_3 = 1-t$

For this point to lie on above sphere, need:

$$1 = (tx)^2 + (ty)^2 + (1-t)^2 = t^2(x^2 + y^2 + 1) - 2t + 1$$

which has two solutions $t=0$ (corresponding to N =north pole)

and $t = \frac{2}{x^2 + y^2 + 1} = \frac{2}{|z|^2 + 1}$, where $z = x+iy$

Plugging this into $x_1 = tx$, $x_2 = ty$, $x_3 = 1-t$ implies the claimed f-las.

Example: For $z = 1+i$, we get $(x_1, x_2, x_3) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

We can also derive the opposite formula: expressing x,y via x_1, x_2, x_3

Lecture #5

Lemma 2: A point (x_1, x_2, x_3) on a unit sphere has stereographic projection $z = x + iy$ with $x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$

As before: $x_1 = tx$, $x_2 = ty$, $x_3 = 1-t$
 Solving $x_3 = 1-t \Rightarrow t = 1-x_3 \Rightarrow x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$

As an important application of the above formula, we have:

Fact: All lines and circles in the complex plane correspond to circles on the Riemann sphere

Indeed, any circle or line in $\mathbb{C}$ -plane is given by
 (1) $A(x^2 + y^2) + Bx + Cy + D = 0$ with $A, B, C, D \in \mathbb{R}$, $A \geq 0$.

(Moreover $A=0$ corresponds to lines, $A>0$ - to circles)

Plugging above f -as $x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$, and multiplying by $(1-x_3)^2$ get:

$$A(\underbrace{x_1^2 + x_2^2}_{=1-x_3^2 = (1-x_3)(1+x_3)}) + Bx_1(1-x_3) + Cx_2(1-x_3) + D(1-x_3)^2 = 0$$

Dividing by $1-x_3$, we get an equation of a plane

$$(2) \quad A(1+x_3) + Bx_1 + Cx_2 + D(1-x_3) = 0$$

Thus, the stereographic projection of (1) satisfies (2) and hence belongs to a circle obtained as the intersection of above plane and the unit sphere. It's clear that each point on that circle arises that way (reversing above argument)

Note: Conversely every circle on the Riemann sphere is mapped under stereographic projection to a line or circle.

By the above fact, an open disk around any point $z_0 \in \mathbb{C}$ will correspond to a cap around its stereographic projection (on the unit sphere) bounded by a circle, like in



Fig. 1



Fig. 2

However, we note a special role of the north pole "N": it doesn't correspond to any point on the complex plane. However, if we consider a small cap around N, like in Fig. 2, then under stereographic projection it clearly corresponds to $\{z \in \mathbb{C} \mid |z| \geq R\}$ for some real R.

① With this perspective in mind, we think of the Riemann sphere as a one-point compactification of the complex plane, where "N = North pole" corresponds to a point " ∞ ". Thus $\mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$ is the extended complex plane.

Let us now present the versions of limits, when either sequence tends to ∞ , or the limit, or both.

Def: a) A sequence $\{z_n\}_{n=1}^{\infty}$ has a limit ∞ , denoted $\lim_{n \rightarrow \infty} z_n = \infty$, if for any real R there is N such that $|z_n| > R$ for all $n > N$.

a') $\lim_{z \rightarrow z_0} f(z) = \infty$ if for any real R there is $\delta > 0$ s.t. $|f(z)| > R$ if $0 < |z - z_0| < \delta$.

b) Likewise, we say $\lim_{z \rightarrow \infty} f(z) = w_0$ if for any $\varepsilon > 0$ there is $R \in \mathbb{R}$ such that $|f(z) - w_0| < \varepsilon$ whenever $|z| > R$.

c) Finally, we say $\lim_{z \rightarrow \infty} f(z) = \infty$ if for any $R \in \mathbb{R}$ there is $K \in \mathbb{R}$ s.t. $|f(z)| > R$ for $|z| > K$.

Ex 4: Show that:

$$a) \lim_{z \rightarrow i} \frac{z^2 + 3z}{z^2 + 1} = \infty$$

$$b) \lim_{z \rightarrow \infty} \frac{2z + 2}{iz + 4} = -2i$$

$$c) \lim_{z \rightarrow \infty} \frac{2z^2 + 2}{iz + 4} = \infty$$

Warning: Unlike in school where you used two different symbols $+\infty, -\infty$ we now have a single " ∞ " when speaking about extended complex plane.