

Lecture #6

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Last week: limits & continuity of functions $f: \mathbb{C} \rightarrow \mathbb{C}$

Today: §2.3-2.4.

Let's start right away with the definition:

Def: Let f be $\mathbb{C}$ -valued function defined in a neighborhood of z_0 (i.e. in some small open disk $D_\varepsilon(z_0)$). The derivative of f at z_0 is

$$f'(z_0) = \frac{df}{dz}(z_0) := \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

↑ when z_0 is fixed we view $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ as a function of Δz

[Note: This looks exactly as the derivative of $\mathbb{R}$ -valued function on $\mathbb{R}$, except that now $\Delta z \rightarrow 0$ means $|\Delta z| \rightarrow 0$ while it can approach "from any direction"]

Example: a) If $f(z) = z$, then $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z) - z_0}{\Delta z} = 1$

b) If $f(z) = z^2$, then $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} (2z_0 + \Delta z) = 2z_0$

c) More generally, if $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ - any polynomial, then $f'(z_0) = n a_n z_0^{n-1} + (n-1) a_{n-1} z_0^{n-2} + \dots + a_1$ (just like in school!)

Ex 1: Describe all points at which the following functions are differentiable:

a) $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = \bar{z}$

b) $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = \operatorname{Re}(z)$

c) $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = \operatorname{Im}(z)$

► a) $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{\overline{z_0 + \Delta z} - \bar{z}_0}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\bar{\Delta z}}{\Delta z}$

Let's write $\Delta z = |\Delta z| \cdot e^{i\theta}$. Then $\frac{\bar{\Delta z}}{\Delta z} = e^{i(-2\theta)}$ - can be any point on unit circle while $|\Delta z| \rightarrow 0 \Rightarrow$ no limit.

[Alternatively, you can just note that if Δz approaches 0 along x-axis, then $\frac{\bar{\Delta z}}{\Delta z} \rightarrow 1$, while if Δz approaches 0 along y-axis, then $\frac{\bar{\Delta z}}{\Delta z} \rightarrow -1$. Therefore, the limit doesn't exist at any point.]

b) Like in above line, if $\Delta z \rightarrow 0$ along x-axis, then $\frac{\operatorname{Re}(z_0 + \Delta z) - \operatorname{Re}(z_0)}{\Delta z} \rightarrow 1$, while if $\Delta z \rightarrow 0$ along y-axis, then $\frac{\operatorname{Re}(z_0 + \Delta z) - \operatorname{Re}(z_0)}{\Delta z} \rightarrow 0$. Hence nowhere differentiable.

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c) Similar argument!

So each of these 3 functions is actually nowhere differentiable!

Following the same arguments as in usual calculus, we obtain:

Proposition: a) If $f(z), g(z)$ are differentiable at z_0 , then:

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

$$(c \cdot f)'(z_0) = c \cdot f'(z_0) \quad \text{for any } c \in \mathbb{C}$$

$$(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$$

b) If $f(z), g(z)$ are differentiable at z_0 and $g(z_0) \neq 0$, then

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

c) If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then

$$\text{"chain rule"}: \frac{d(f \circ g)}{dz}(z_0) = f'(g(z_0)) \cdot g'(z_0)$$

Ex2: Compute the derivative of $f(z) = \left(\frac{z^3+z+1}{z^2+4}\right)^{2025}$

$$\Rightarrow f'(z_0) = 2025 \cdot \left(\frac{z_0^3+z_0+1}{z_0^2+4}\right)^{2024} \cdot \frac{(3z_0^2+1)(z_0^2+4) - (z_0^3+z_0+1) \cdot 2z_0}{(z_0^2+4)^2}$$

Q: What about complex exponent $z \mapsto e^z$: is it everywhere differentiable?

A: We shall now discuss a general criteria that will imply answer "Yes"

Terminology:

a) A $\mathbb{C}$ -valued function $f(z)$ is called analytic on an open set Ω of a complex plane if $f'(z)$ exists for every $z_0 \in \Omega$.

b) A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is called entire if it is analytic on all $\mathbb{C}$.

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Theorem 1: If a function $f(z) = u(x, y) + i \cdot v(x, y)$ with $z = x + iy$ is differentiable at a point $z_0 = x_0 + iy_0$, then the following holds:

$$\text{Cauchy-Riemann equations} \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

In particular, if $f(z)$ is analytic on an open set Ω , then these Cauchy-Riemann (CR) equations hold at each point of Ω .

Proof

Let's evaluate $\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ by first letting Δz approach 0 along real axis, i.e. we take $\Delta z = \varepsilon + 0 \cdot i$ with $\varepsilon \rightarrow 0$. Then, we get:

$$\begin{aligned} f'(z_0) &= \lim_{\varepsilon \rightarrow 0} \frac{u(x_0 + \varepsilon, y_0) + i \cdot v(x_0 + \varepsilon, y_0) - u(x_0, y_0) - i \cdot v(x_0, y_0)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{u(x_0 + \varepsilon, y_0) - u(x_0, y_0)}{\varepsilon} + i \cdot \lim_{\varepsilon \rightarrow 0} \frac{v(x_0 + \varepsilon, y_0) - v(x_0, y_0)}{\varepsilon} = u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Let's now evaluate above limit when $\Delta z = i\varepsilon$ with $\varepsilon \rightarrow 0$ being real:

$$\begin{aligned} f'(z_0) &= \lim_{\varepsilon \rightarrow 0} \frac{u(x_0, y_0 + \varepsilon) + i \cdot v(x_0, y_0 + \varepsilon) - u(x_0, y_0) - i \cdot v(x_0, y_0)}{i\varepsilon} \\ &= \frac{1}{i} \lim_{\varepsilon \rightarrow 0} \frac{u(x_0, y_0 + \varepsilon) - u(x_0, y_0)}{\varepsilon} + \lim_{\varepsilon \rightarrow 0} \frac{v(x_0, y_0 + \varepsilon) - v(x_0, y_0)}{\varepsilon} \\ &= -i \cdot u_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

So: $u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) = v_y(x_0, y_0) - i \cdot u_y(x_0, y_0)$ which implies the result $\square$

To illustrate, let's consider $f(z) = \operatorname{Re} z$ from Ex 1(b). Then $\begin{cases} u(x, y) = x \\ v(x, y) = 0 \end{cases}$ and $u_x(x_0, y_0) = 1$ (for any $x_0, y_0 \in \mathbb{R}$) while $v_y(x_0, y_0) = 0 \neq 1$, hence by Thm 1 it's nowhere differentiable. Same can be applied to Ex 1(a, c).

Q: Is the CR equations also sufficient for f to be differentiable at $z_0 = x_0 + iy_0$?

A: Under some mild conditions, "Yes"!

Theorem 2: Let $f(z) = u(x, y) + i \cdot v(x, y)$ with $z = x + iy$ be a function defined in some small neighborhood Ω of $z_0 = x_0 + iy_0$, i.e. in small disk $D_\epsilon(z_0)$. Assume that all four partial derivatives u_x, u_y, v_x, v_y exist in Ω , continuous at z_0 and satisfy Cauchy-Riemann eq-s at z_0 . Then $f'(z_0)$ exists!

↑ the interested students can find the proof in pages 74-75 of textbook
called u, v are " C^1 -smooth on Ω "

Upshot: If u_x, u_y, v_x, v_y are continuous on an open set Ω , then $f: \Omega \rightarrow \mathbb{C}$ is analytic $\Leftrightarrow$ Cauchy-Riemann equations hold at each point of Ω .

Example: Show that $f(z) = e^z$ is an entire function

▮ If $z = x + iy$, then $f(z) = e^{x+iy} = e^x(\cos y + i \sin y) \Rightarrow \begin{cases} u(x, y) = e^x \cos y \\ v(x, y) = e^x \sin y \end{cases}$
 $u_x = e^x \cos y = v_y$ & $u_y = -e^x \sin y = -v_x$
 All partial derivatives u_x, u_y, v_x, v_y are continuous on $\mathbb{C}$ $\Rightarrow e^z$ -entire.

Exercise: Describe all entire $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $\text{Im } f(z) = \text{const}$

▮ $\text{Im } f(z) = v(x, y) \Rightarrow v_x = 0 = v_y \xRightarrow{\text{CR}} u_x = 0 = u_y \Rightarrow u = \text{const} \Rightarrow f$ -constant function

Exercise: Prove that if f -analytic in domain Ω and $|f(z)| = \text{constant}$, then $f(z) = \text{constant}$
 (think at home (will discuss next class))

Book: Note that §2.3 of book has some non-rigorous blurb on "admissible functions", but what that really means is that one wants to study functions intrinsically written via z , not $\bar{z}$, $\text{Re } z$, $\text{Im } z$.
 → Next time: §2.5