Lecture #7

Due to the Survey' results, let's practice more today, leaving §2.5 to the end.

Ex1: (True/False) Any $f:\mathbb{C}\to\mathbb{C}$ differentiable at z_0 is continuous at z_0

▸ True : As $g(z) := \dfrac{f(z)-f(z_0)}{z-z_0} \xrightarrow[z\to z_0]{} a\in\mathbb{C} \Rightarrow f(z)-f(z_0) = a(z-z_0) + h(z)(z-z_0)$ with $h(z)\xrightarrow[z\to z_0]{} 0$

Thus, $|f(z)-f(z_0)| \underset{\substack{\uparrow\\ \text{triangular inequality}}}{\leq} |a|\cdot|z-z_0| + |h(z)|\cdot|z-z_0| \longrightarrow 0$ as $z\to z_0$ ■

Ex2 : Compute $\displaystyle\lim_{z\to i}\frac{1+z^6}{1+z^{10}}$

[Hint: As in usual calculus, we can apply L'Hopital rule which says that if $f(z)$, $g(z)$ are analytic at z_0, with $f(z_0)=0=g(z_0)$, $g'(z_0)\neq 0$, then:

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{f'(z_0)}{g'(z_0)}$$]

▸ For $f(z)=1+z^6$, $g(z)=1+z^{10}$, note that $f(i)=0=g(i)$, while $f'(i)=6i^5$, $g'(i)=10i^9$

So: $\displaystyle\lim_{z\to i}\frac{f(z)}{g(z)} = \frac{6i^5}{10i^9} = \frac{3}{5}$ as $i^4=1$ ■

Ex3: At which points is $z\mapsto |z|^2 = f(z)$ differentiable?

▸ Proof #1 (from definition)

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{(z_0+\Delta z)(\overline{z_0}+\overline{\Delta z})-z_0\cdot\overline{z_0}}{\Delta z} = \lim_{\Delta z\to 0}\frac{\overline{z_0}\cdot\Delta z + z_0\cdot\overline{\Delta z}+\Delta z\cdot\overline{\Delta z}}{\Delta z}$$

$$= \overline{z_0} + \lim_{\Delta z\to 0} z_0\cdot\frac{\overline{\Delta z}}{\Delta z}$$

If $z_0=0$, then we just get $\overline{z_0}=0 \Rightarrow f'(0)$ is well-defined and equals 0.

If $z_0\neq 0$, then if $f'(z_0)$ existed, we would also have $\displaystyle\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$ which we showed last time doesn't exist.

Answer: Only at 0.

Proof #2 (Cauchy-Riemann $\overset{\text{Much faster!}}{=}$ CR) Just need to find point where $\begin{cases}u_x=v_y\\ u_y=-v_x\end{cases}$

But $v(x,y)=0 \Rightarrow v_x=0=v_y$, while $u(x,y)=x^2+y^2 \Rightarrow u_x=2x$, $u_y=2y$

Hence CR holds only at $x=0, y=0 \Rightarrow$ at $z=0$ ■

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Ex 4: Is $f(z) = 3x^2+2x-3y^2-1+(6xy+2y)i$ an entire function?

If so, express it solely via z.

- By definition $f(z)$ - entire $\Leftrightarrow$ f is differentiable at each point

For the latter it suffices to check CR at each point.

$$\left.\begin{array}{ll} u(x,y) = 3x^2+2x-3y^2-1 & \Rightarrow u_x = 6x+2,\ u_y = -6y \\ v(x,y) = 6xy+2y & \Rightarrow v_x = 6y,\ v_y = 6x+2 \end{array}\right\} \Rightarrow \text{CR holds!}$$

So: yes $f(z)$ is entire.

- Looking carefully at $f(z)$ it is easy to recognize that $f(z) = 3z^2+2z-1$ where $z = x+yi$

But: in general, if you didn't see this right away, just plug $x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$. Let's do it in above example:

$$f(z) = 3\left(\frac{z+\bar z}{2}\right)^2 + 2\left(\frac{z+\bar z}{2}\right) - 3\left(\frac{z-\bar z}{2i}\right)^2 - 1 + \left(6\cdot\frac{z+\bar z}{2}\cdot\frac{z-\bar z}{2i} + 2\cdot\frac{z-\bar z}{2i}\right)i$$

$$= \frac{3}{4}(z^2+\bar z^2+2z\bar z) + (z+\bar z) + \frac{3}{4}(z^2+\bar z^2-2z\bar z) - 1 + \frac{6}{4}(z+\bar z)(z-\bar z) + (z-\bar z)$$

$$= \frac{6}{4}z^2 + \frac{6}{4}\bar z^2 + z + \bar z - 1 + \frac{6}{4}z^2 - \frac{6}{4}\bar z^2 + z - \bar z = 3z^2+2z-1$$

Note: Once you know that $f(z)$ is entire and plug $x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$, you are guaranteed that after all cancelations there will be no $\bar z$!

! Comment regarding some HWK 2 problems from §1.7, 2.1 that were asking to geometrically describe $w = f(z)$ as a transformation of the Riemann sphere:

- if you don't know geometric answer, there is only one way to do:

$$\left.\begin{array}{ll} z \mapsto (x_1,x_2,x_3) & \text{via Lemma 1 of Lecture 5} \\ w \mapsto (x_1',x_2',x_3') & \text{—//—} \end{array}\right\} \text{try to relate } (x_1',x_2',x_3') \text{ to } (x_1,x_2,x_3)$$

- in #15 from p. 58 it will actually be much faster to do opposite way:

geom. description

$$\left.\begin{array}{ll} (x_1,x_2,x_3) \mapsto z & \text{via Lemma 2 of Lecture 5} \\ (x_1',x_2',x_3') \mapsto w & \text{—//—} \end{array}\right\} \text{verify these } z, w \text{ are related by stated f-la}$$

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The essence of Cauchy-Riemann eqs from [Thm 1, Thm 2 of Lecture 6] is as follows. Given $u, v: \mathbb{R}^2 \to \mathbb{R}$ such that u_x, u_y, v_x, v_y exist and are continuous:

$$\boxed{\begin{array}{l} f(z=x+iy) := u(x,y) + i \cdot v(x,y) \\ \text{is differentiable at } z_0 = x_0 + iy_0 \end{array} \iff \underline{\underline{\text{CR}}} \text{ holds } \begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}}$$

Note: The matrix consisting of first partial derivatives now takes the form

$$\underbrace{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}_{\uparrow \text{called the Jacobian}} \overset{CR}{=\!=} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which corresponds to a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which is a composition of rotation and scaling

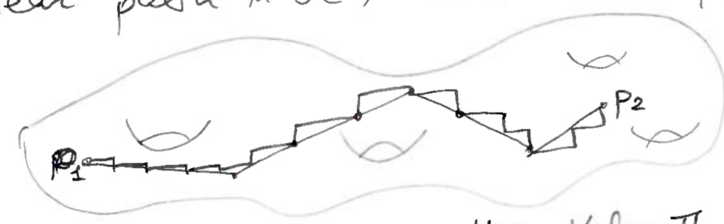
which corresponds to $(A+Bi)\cdot$ as a map $\mathbb{C} \to \mathbb{C}$

Lemma: If $f(z)$ - analytic in a domain Ω, $f'(z) = 0$ in $\Omega \Rightarrow f(z) = \text{const}$

The equality $f'(z_0) = u_x(x_0,y_0) + i \cdot v_x(x_0,y_0) = -i \cdot u_y(x_0,y_0) + v_y(x_0,y_0)$ from Lecture 6 imply:

$u_x = 0 = u_y$ in Ω and similarly $v_x = 0 = v_y$ in Ω

As Ω is connected, any two points can be connected by a piece-wise linear path in Ω, which can be further approximated by a piece-wise linear path with each segment being horizontal or vertical.



follows from Mean Value Theorem.

$$\left.\begin{array}{l} \text{As } u_x = 0 \Rightarrow u \text{ is constant on all horizontal segments} \\ \phantom{\text{As }} u_y = 0 \Rightarrow u \quad -\text{''}- \quad \text{vertical segments} \end{array}\right\} \Rightarrow u(p_1) = u(p_2)$$

Similarly: $v(p_1) = v(p_2) \Rightarrow f$ is constant

Corollary: a) If f, g - analytic in domain and $f' = g'$ in $\Omega \Rightarrow f - g = \text{constant}$

b) If f - analytic and $\operatorname{Im} f$ is constant $\Rightarrow f$ - constant

c) If f - analytic and $\operatorname{Re} f$ is constant $\Rightarrow f$ - constant

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Ex 5: Show that if $f(z)$ is analytic in domain and $|f(z)| = \text{constant}$, then actually $f(z) = \text{constant}$.

- If $|f(z)| = 0 \Rightarrow$ clear as $f(z) = 0$ for all z.
- Assume $|f(z)| = c > 0$. Write $f(x+iy) = u(x,y) + i\cdot v(x,y)$.

Then $u^2(x,y) + v^2(x,y) = c^2$. Apply ∂_x or ∂_y to get:

$$\begin{cases} 2\cdot u\cdot u_x + 2v\cdot v_x = 0 \\ 2\cdot u\cdot u_y + 2v\cdot v_y = 0 \end{cases}$$

← each u, v, u_x, u_y, v_x, v_y is evaluated at the given point

But f - analytic $\overset{CR}{\Rightarrow} u_x = v_y,\ u_y = -v_x$. Hence, we get:

$$\left.\begin{array}{l} \begin{cases} u\cdot v_y + v\cdot v_x = 0 \\ -u\cdot v_x + v\cdot v_y = 0 \end{cases} \Rightarrow \begin{pmatrix} v & u \\ -u & v \end{pmatrix}\cdot\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{However, } \det\begin{pmatrix} v & u \\ -u & v \end{pmatrix} = u^2 + v^2 = c^2 \neq 0 \end{array}\right\} \Rightarrow v_x = v_y = 0 \overset{CR}{\Rightarrow} u_x = u_y = 0.$$

Hence, by the same argument as in the proof of lemma above:

$u = \text{const},\ v = \text{const} \Rightarrow f$ - constant

§2.5 Harmonic functions

Def: A function $g: \underbrace{\Omega}_{\text{domain in } \mathbb{R}^2} \rightarrow \mathbb{R}$ is called harmonic if it satisfies the following equation $\underbrace{\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0}$

Laplace's equation, often written $\nabla^2 g = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Theorem: If $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic in a domain $\Omega \subseteq \mathbb{C}$ then $\text{Re}(f)$, $\text{Im}(f)$ are both harmonic functions in Ω

Write $f(x+iy) = u(x,y) + i\cdot v(x,y)$. We shall see later: $u(x,y)$, $v(x,y)$ admit continuous partial derivatives of any order!

(this is a surprising fact, very different from usual calculus)

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Continuation of the proof

C–R : $u_x = v_y \xrightarrow{\text{apply } \partial_x} u_{xx} = v_{yx}$

$u_y = -v_x \xrightarrow[\text{apply } \partial_y]{} u_{yy} = -v_{xy}$

$\Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ as both v_{xy}, v_{yx} are continuous

$\Downarrow$

$\boxed{u\text{-harmonic}}$

C–R : $u_x = v_y \xrightarrow{\text{apply } \partial_y} v_{yy} = u_{xy}$

$-u_y = v_x \xrightarrow[\text{apply } \partial_x]{} v_{xx} = -u_{yx}$

$\Rightarrow v_{xx} + v_{yy} = u_{xy} - u_{yx} = 0$

$\Downarrow$

$\boxed{v\text{-harmonic}}$

Next time : Finish §2.5 & start §3.1