Lecture #8

Last time we ended with the following result:

Theorem 1: If $f(z=x+iy)=u(x,y)+i\cdot v(x,y)$ is an analytic function in a domain Ω then both $u,v:\Omega\to\mathbb{R}$ are harmonic

Note: One can rephrase this as follows: if $u,v:\Omega\to\mathbb{R}$ are C^2-smooth and satisfy CR in Ω, then u,v are harmonic in Ω.

In fact, while last time we saw one needs this C^2-smoothness, it follows from:

Surprising Fact: If $f:\Omega\to\mathbb{C}$ is analytic, then $f':\mathbb{C}\to\mathbb{C}$ and all $f^{(n)}:\mathbb{C}\to\mathbb{C}$ are

Q: Given a C^2-smooth harmonic $u:\Omega\to\mathbb{R}$, can we find analytic $f:\Omega\to\mathbb{C}$ with $\operatorname{Re}(f)=u$, i.e. can we find another harmonic $v:\Omega\to\mathbb{R}$ s.t. (u,v)-CR

Let's start with an example.

Ex 1: Let $u(x,y)=x^2-y^2+3xy+1$. Verify it is harmonic. Find all $v(x,y)$ such that u,v satisfy CR. Verify v is harmonic

- $u_{xx}=2,\ u_{yy}=-2 \Rightarrow u_{xx}+u_{yy}=0 \Rightarrow u$ - harmonic.
- Look for $v(x,y)$ such that $\begin{cases} v_x=-u_y=2y-3x \\ v_y=u_x=2x+3y. \end{cases}$

Step 1: Solve $v_x=2y-3x$ for any fixed y: $v(x,y)=\int(2y-3x)dx=2xy-\frac{3}{2}x^2+A(y)$

Step 2: Solve $2x+3y=v_y=2x+A'(y) \Rightarrow A(y)=\frac{3}{2}y^2+\underbrace{B}_{\text{constant}}$

$$\Rightarrow \boxed{v(x,y)=2xy-\frac{3}{2}x^2+\frac{3}{2}y^2+B}$$

- $v_{xx}=-3,\ v_{yy}=3 \Rightarrow v$ - harmonic

Note: It is clear that if $v,\tilde{v}$ - two possible solutions, then $\partial_x(v-\tilde{v})=0=\partial_y(v-\tilde{v})$ hence $v,\tilde{v}$ differ by adding a constant (given Ω - domain)

Rmk: In Step 2 there will never be x after cancelations unless you made error

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In fact, the general result from calculus asserts that if Ω is an open disk, a rectangle, or more generally a domain without "holes", then $\begin{cases} v_x = \varphi(x,y) \\ v_y = \psi(x,y) \end{cases}$ admits a solution iff $\varphi_y = \psi_x$. But in our case of $\varphi = -u_y$, $\psi = u_x$ (by CR), this reduces to $u_{xx} + u_{yy} = 0$ which holds as u - harmonic!

(Also: $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0 \Rightarrow v$ - harmonic too)

$\Downarrow$

Theorem 2: On any domain without "holes" (known as simply-connected) for any harmonic u there is another harmonic v such that u, v satisfy CR and thus give rise to analytic $f(z) = u + iv$

Def: Such v is called harmonic conjugate of u (and is unique up to adding a constant)

A classical counterexample to such statement for not simply-connected domains is (we shall return to it in a few lectures):

Ex2*: Verify that $u(x,y) = \ln\sqrt{x^2+y^2} : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}$ is harmonic, BUT there is no analytic $f(z): \mathbb{C}\setminus\{0\} \to \mathbb{C}$ with $\mathrm{Re}(f) = u$.

Let us now solve the following exercise (a bit tricky!):

Ex3: Verify that $u(x,y) = r^n \cos(n\theta)$, where n - integer and r, θ - polar coor., is a harmonic function, and find its harmonic conjugate

One can recall that $r^n \cos(n\theta) = \mathrm{Re}(z^n)$ with $z = re^{i\theta}$ by DeMoivre's thm. But $z \mapsto z^n$ is analytic on $\mathbb{C}\setminus\{0\}$. Hence, $\mathrm{Re}(z^n)$ - harmonic and moreover $\mathrm{Im}(z^n) = r^n \sin(n\theta)$ is a harmonic conjugate. ∎

Rmk: In the next homework you will see that Laplace's equation in polar coordinates takes the form $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)u = 0$. Hence, the first part of Ex3 can be easily verified directly

Ex3': Find harmonic conjugate of $r^n \sin(n\theta)$.

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We shall conclude §2.5 by discussing level curves $u(x,y) = A \in \mathbb{R}$ and $v(x,y) = B \in \mathbb{R}$. Recall that the gradient vector field $\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ is orthogonal to the level curves of $u : \Omega \to \mathbb{R}$. Likewise, $\nabla v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ is orthogonal to the level curves of $v : \Omega \to \mathbb{R}$. But

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y \overset{CR}{=} -v_y v_x + v_x v_y = 0 \Rightarrow \nabla u \perp \nabla v \text{ at each point}$$

$$\Downarrow$$

Fact: Level curves of u and v are always orthogonal

here v is harmonic conjugate of u

§3.1 Polynomials and rational functions

Recall that polynomials in 1 variable are $p(z) = a_n z^n + a_{n-1} z^{n-1} + \_ + a_1 z + a_0$. If $a_n \neq 0$, then degree of $p(z)$ is $\deg(p) = n$. As stated in 1^{st} class:

Theorem 3 (Fundamental Theorem of Algebra): Any non-constant polynomial with complex coefficients has at least one root in $\mathbb{C}$. Therefore, each polynomial can be factorized as $p(z) = a_n (z - z_1) \dots (z - z_n)$ for some $z_1, z_2, \_, z_n \in \mathbb{C}$

Ex 4: Prove that any nonconstant polynomial with real coefficients can be factored into product of linear and quadratic polynom-s with real coefficients.

▶ By HwK #1: if $p(z_0) = 0 \Rightarrow p(\bar{z}_0) = 0$. Thus, if $z_0 \in \mathbb{C}$ - is a root of $p(z)$ which is not real, then $\bar{z}_0$ - also a root. However:

$(z - z_0)(z - \bar{z}_0) = z^2 - 2\,\mathrm{Re}(z_0) \cdot z + |z_0|^2$ which has real coeff-s.

The result follows! ◻

Note: If some $z_0 \in \mathbb{C}$ appears k times among $\{z_1, \_, z_n\}$ in Thm 3, then $k = $ multiplicity of root z_0.

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Ex 5: Write $1+z+z^2+z^3+z^4+z^5+z^6$ in the factored form over $\mathbb{C}$

$$1+z+\_\_+z^6 = \frac{1-z^7}{1-z} = \frac{z^7-1}{z-1}$$

By Lecture 7, we know $z^7-1 = (z-1)(z-\omega_7)(z-\omega_7^2)(z-\omega_7^3)(z-\omega_7^4)(z-\omega_7^5)(z-\omega_7^6)$

where $\omega_7 = e^{i\cdot\frac{2\pi}{7}} = \cos\left(\frac{2\pi}{7}\right)+i\sin\left(\frac{2\pi}{7}\right)$

So: $1+z+z^2+z^3+z^4+z^5+z^6 = (z-\omega_7)(z-\omega_7^2)(z-\omega_7^3)(z-\omega_7^4)(z-\omega_7^5)(z-\omega_7^6)$ ∎

Ex 5': Factorize $1+z+z^2+z^3+z^4+z^5+z^6$ into linear & quadratic over $\mathbb{R}$.

Following solution of Ex 4, note that $\omega_7^6=\overline{\omega_7}$, $\omega_7^5=\overline{\omega_7^2}$, $\omega_7^4=\overline{\omega_7^3}$

Moreover, $\omega_7^k+\overline{\omega_7^k} = 2\cos\frac{2\pi k}{7}$, $\omega_7^k\cdot\overline{\omega_7^k}=1$

So: $1+z+\dots+z^6 = \left(z^2-2\cos\frac{2\pi}{7}+1\right)\left(z^2-2\cos\frac{4\pi}{7}+1\right)\left(z^2-2\cos\frac{6\pi}{7}+1\right)$ ∎

Next time: §3.1–3.2

(! recall partial fraction decomposition before class)