

Lecture #9 (§ 3.1-3.2)

Ex1: Is $|z|^2 + e^{z^2}$ an entire function?

- No. If $f(z) = |z|^2 + e^{z^2}$ was entire, then $|z|^2 = f(z) - e^{z^2}$ would also be. But $|z|^2 = z \cdot \bar{z}$, and so $\bar{z} = \frac{f(z) - e^{z^2}}{z}$ would be analytic on $\mathbb{C} \setminus \{0\}$. But we know that $\bar{z}$ is nowhere differentiable, a contradiction. ■

Ex2: Find poles and their multiplicities of $R(z) = \frac{z^4 - 3z^2 + 2}{(z-1)(z^2+4)^2}$.

- Note that $z^4 - 3z^2 + 2 = (z^2-1)(z^2-2) = (z-1)(z+1)(z-\sqrt{2})(z+\sqrt{2})$
 $(z-1)(z^2+4)^2 = (z-1)(z-2i)^2(z+2i)^2$ } $\Rightarrow$

$$\Rightarrow R(z) = \frac{(z+1)(z-\sqrt{2})(z+\sqrt{2})}{(z-2i)^2(z+2i)^2} \text{ has poles at } \pm 2i, \text{ both of multiplicity 2. } \blacksquare$$

Ex3: Decompose $z^2 + 3z + 2$ in terms of powers of $z+1$.

- Solution 1: Want $z^2 + 3z + 2 = a_n(z+1)^n + a_{n-1}(z+1)^{n-1} + \dots + a_1(z+1) + a_0$, $a_n \neq 0$

Comparing max powers of z we see that $n=2$ & $a_n=1$. Thus:

$$z^2 + 3z + 2 = (z+1)^2 + a_1(z+1) + a_0 \Rightarrow z+1 = a_1(z+1) + a_0 \Rightarrow a_1=1, a_0=0. \text{ So:}$$

$$z^2 + 3z + 2 = 1 \cdot (z+1)^2 + 1 \cdot (z+1)$$

Note: The fact $a_0=0$ actually implies that -1 is a root!

Solution 2: If we set $w := z+1$, then $z = w-1$. Thus, we get answer by plugging $w-1$ instead of z into original polynomial:

$$z^2 + 3z + 2 = (w-1)^2 + 3(w-1) + 2 = w^2 + w \stackrel{w=z+1}{=} (z+1)^2 + (z+1)$$

Solution 3: Finally, one could also apply the Taylor form expansion

If $p(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n$, then

$$a_0 = p(z_0)$$

$$1! \cdot a_1 = p'(z_0)$$

$$2! \cdot a_2 = p''(z_0)$$

$\vdots$

$$n! \cdot a_n = p^{(n)}(z_0)$$

$$\Rightarrow a_k = \frac{1}{k!} p^{(k)}(z_0) \text{ with } k! = 1 \cdot 2 \cdot \dots \cdot k$$

we will be recalling Taylor series later in the course!

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[Note: For polynomial $p(z)$ of degree n , the Taylor series truncates, i.e. has finite number of nonzero terms as $p^{(n)}(z) = 0$.

Let's now switch to rational functions and recall the principle of partial fraction decomposition, used e.g. to integrate rational f.s.

Example: $\frac{x-2}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$ for some unique constants A, B yet to be found.

Approach 1 (usually taught in school/calculus class): reduce to linear eqs

$$x-2 = A(x-3) + B \cdot x \Leftrightarrow \begin{cases} A+B=1 \\ -3A=-2 \end{cases}$$

Approach 2: Find constants A, B by specializing x !

Plug $x=0$ into $x-2 = A(x-3) + Bx$ to find $-2 = -3A \Rightarrow A = 2/3$

Plug $x=3$ — " — to find $1 = 3B \Rightarrow B = 1/3$

Ex 4: Decompose $\frac{2z+i}{z^3+z}$ into partial fractions.

► $z^3+z = z(z-i)(z+i)$, hence, we are looking for decomposition

$$\frac{2z+i}{z^3+z} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i} \quad \text{goal: find } A, B, C$$

Bridging to common denominator find:

$$2z+i = A(z-i)(z+i) + Bz(z+i) + Cz(z-i)$$

Plug $z=0$: $i = A$

Plug $z=i$: $3i = B \cdot i \cdot 2i = -2B \Rightarrow B = -\frac{3i}{2}$

Plug $z=-i$: $-i = C \cdot (-i) \cdot (-2i) = -2C \Rightarrow C = \frac{i}{2}$

$$\left\{ \frac{i}{z} + \frac{-\frac{3i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i} \right\}$$

Ex 4': Same for $\frac{2z^4+2z^2+2z+i}{z^3+z}$

► Note $2z^4+2z^2+2z+i = 2z \cdot (z^3+z) + 2z+i \Rightarrow$

$$\left\{ 2z + \frac{i}{z} + \frac{-\frac{3i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i} \right\}$$

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The above examples illustrate two key general ideas:

- 1) if denominator has only multiplicity 1 roots, then all coefficients in partial fraction decomposition are obtained by evaluation at these roots (after bringing to the common denominator)
- 2) if $R(z) = \frac{p(z)}{q(z)}$ and $\deg p \geq \deg q$, then you first write $p(z) = q(z) \cdot \alpha(z) + \bar{p}(z)$ with $\deg \bar{p} < \deg q$ ("long division")

Theorem: If $R(z) = \frac{p(z)}{b_n(z-\zeta_1)^{d_1} \dots (z-\zeta_r)^{d_r}}$ is a rational function with the denominator $q(z) = b_n(z-\zeta_1)^{d_1} \dots (z-\zeta_r)^{d_r}$ having roots $\zeta_1, \dots, \zeta_r$ of multiplicities $d_1, \dots, d_r$, and we assume $\deg p < \deg q = d_1 + \dots + d_r$, then $R(z)$ admits the following partial fraction decomposition:

$$R(z) = \frac{A_0^{(1)}}{(z-\zeta_1)^{d_1}} + \frac{A_1^{(1)}}{(z-\zeta_1)^{d_1-1}} + \dots + \frac{A_{d_1-1}^{(1)}}{(z-\zeta_1)} \\ + \frac{A_0^{(2)}}{(z-\zeta_2)^{d_2}} + \frac{A_1^{(2)}}{(z-\zeta_2)^{d_2-1}} + \dots + \frac{A_{d_2-1}^{(2)}}{(z-\zeta_2)} \\ + \dots + \frac{A_{d_r-1}^{(r)}}{(z-\zeta_r)}$$

with

$$A_s^{(j)} = \lim_{z \rightarrow \zeta_j} \frac{1}{s!} \frac{d^s}{dz^s} \left((z-\zeta_j)^{d_j} \cdot R(z) \right)$$

↑ Read the proof in p. 107 of your textbook.

Note: 1) $\frac{A_0^{(j)}}{(z-\zeta_j)^{d_j}} + \dots + \frac{A_{d_j-1}^{(j)}}{(z-\zeta_j)} = \frac{A_0^{(j)} + A_1^{(j)}(z-\zeta_j) + \dots + A_{d_j-1}^{(j)}(z-\zeta_j)^{d_j-1}}{(z-\zeta_j)^{d_j}}$

2) if $\deg p \geq d_1 + \dots + d_r$, we first apply division algorithm as in 2) above.

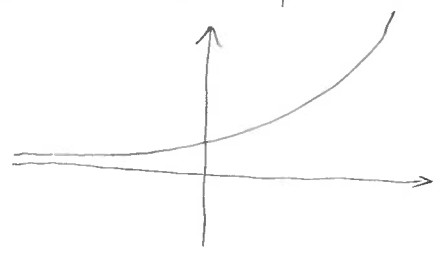
Note: Don't worry too much about long formula in theorem as it will be rarely used.

§3.2

Recall that in Week 1, we already introduced $e^z: \mathbb{C} \rightarrow \mathbb{C}$ via

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

Know that e^z - entire function and $\frac{d}{dz} e^z = e^z$. But in contrast to usual calculus, where $\exp: \mathbb{R} \rightarrow \mathbb{R}$ has a graph



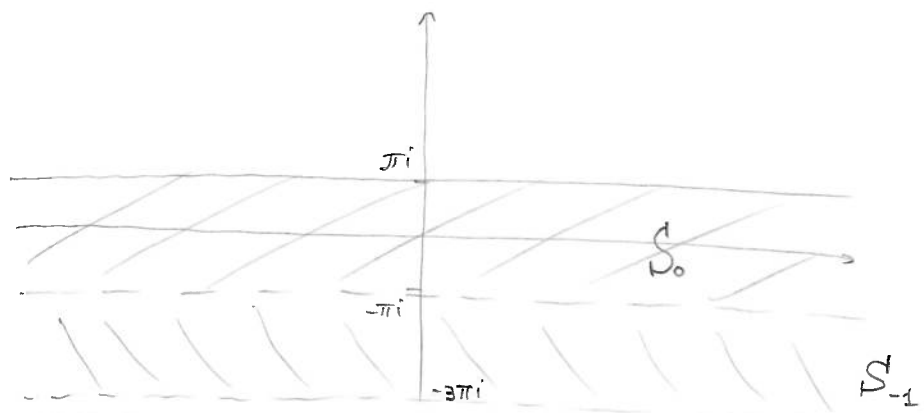
and is clearly one-to-one, this is no longer true now.

$$\text{Lemma: } e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2\pi i k \text{ with } k \in \mathbb{Z}$$

$$\Rightarrow e^{z_1} = e^{z_2} \Rightarrow e^{z_1 - z_2} = 1.$$

$$\text{Let } z_1 - z_2 = x + iy. \text{ Then } e^x \cdot e^{iy} = 1 \Rightarrow x = 0, y = 2\pi k \text{ with } k \in \mathbb{Z}$$

Thus, the complex exponent is a periodic function with period $2\pi i$



$$\text{Let } S_n = \{x + iy \mid x \in \mathbb{R}, (2n-1)\pi < y \leq (2n+1)\pi\}.$$

- Then:
- 1) $e^z: S_n \rightarrow \mathbb{C}$ is one-to-one
 - 2) Range of e^z on S_n is $\mathbb{C} \setminus \{0\}$

For this reason S_n is called "fundamental region" of the exponent map.

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Recall that we already related exponent to trigonometric functions

$$\left. \begin{aligned} e^{i\varphi} &= \cos \varphi + i \sin \varphi \\ e^{-i\varphi} &= \cos \varphi - i \sin \varphi \end{aligned} \right\} \xrightarrow{\varphi \in \mathbb{R}} \boxed{\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \quad \& \quad \sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}}$$

However, we can now extend $\cos, \sin$ to entire functions! (our first example of so-called "analytic extension")

Def: For any $z \in \mathbb{C}$, we define $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

True/False: $\cos^2 + \sin^2 = 1$.

True: $\cos^2 z + \sin^2 z = \frac{e^{2iz} + e^{-2iz} + 2}{4} - \frac{e^{2iz} + e^{-2iz} - 2}{4} = 1$ $\square$

True/False: $|\cos z|$ & $|\sin z| \leq 1$

False: e.g. take $z = -10i \Rightarrow \cos(-10i) = \frac{e^{10} + e^{-10}}{2} \gg 1$

$$|\sin(-10i)| = \left| \frac{e^{10} - e^{-10}}{2i} \right| \gg 1 \quad \square$$

Finally: can also define all other trigonometric f-s for $z \in \mathbb{C}$:

$$\boxed{\tan(z) = \frac{\sin(z)}{\cos(z)}, \cot(z) = \frac{\cos(z)}{\sin(z)}, \sec(z) = \frac{1}{\cos(z)}, \csc(z) = \frac{1}{\sin(z)}}$$

Note: All algebraic equalities on trigonometric functions still hold over $\mathbb{C}$, see formulas (6)-(11) from p. 113 of the textbook. It's a good exercise to check them, if you want extra practice.

Next time: §3.2-3.3.