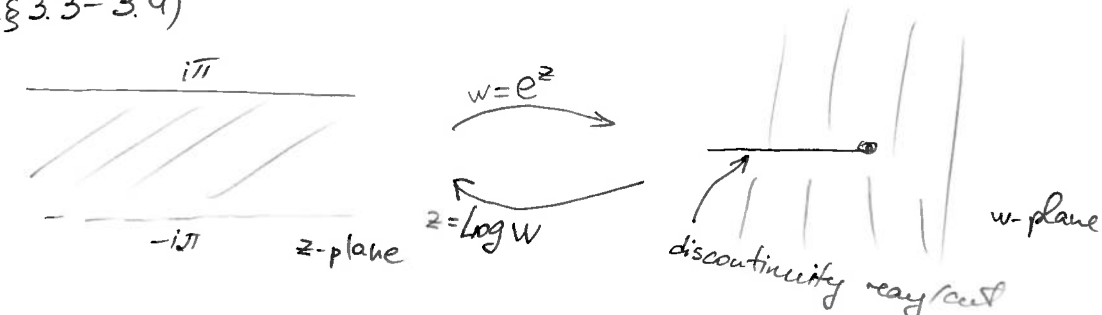
Lecture #11 (§3.3–3.4)

Last time:



$\boxed{\text{Theorem 1: } \mathrm{Log}\, z \text{ is analytic in } D^* = \mathbb{C}\setminus\mathbb{R}_{\leq 0}}$ ($\mathbb{R}_{\leq 0}$ must be excluded as $\mathrm{Log}\, z$ is not even continuous there)

▶ Recall from Homework that Cauchy–Riemann in polar coordinates is:

$$\boxed{\begin{cases} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{cases}} \quad \text{where} \quad u(r,\theta) + i\cdot v(r,\theta) = \underbrace{f}_{\mathbb{C}\text{-valued function whose analyticity we check.}}(re^{i\theta})$$

In our case: $\mathrm{Log}\, z = \overbrace{\mathrm{Log}\,|z|}^{=\text{same as } \ln|z|} + i\cdot \mathrm{Arg}\, z$

$\Downarrow$

$$\begin{cases} u(r,\theta) = \mathrm{Log}\, r = \ln r \\ v(r,\theta) = \theta \text{ with } \theta \in (-\pi, \pi] \end{cases} \Rightarrow \begin{cases} u_r = \frac{1}{r} = \frac{1}{r} v_\theta \\ v_r = 0 = -\frac{1}{r} u_\theta \end{cases} \quad \checkmark$$

∎

[**Note**: The same clearly holds for $\mathrm{Log}_\tau z$ with D^* being rather replaced by $D^*_\tau = \mathbb{C}\setminus\{\text{ray } \theta=\tau\}$

$\boxed{\text{Theorem 2: } \frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z} \text{ in } D^* \quad \left(\text{and similarly } \frac{d}{dz}\mathrm{Log}_\tau z = \frac{1}{z} \text{ in } D^*_\tau\right)}$

▶ Apply $\frac{d}{dz}$ to the equality $z = e^{\mathrm{Log}\, z}$ to get $1 = e^{\mathrm{Log}\, z}\cdot \frac{d}{dz}\mathrm{Log}\, z = z\cdot(\mathrm{Log}\, z)'$

$\Rightarrow (\mathrm{Log}\, z)' = \frac{1}{z}$

∎

[**Rmk**: Using chain rule we see that $\underbrace{\begin{pmatrix} u_r & \frac{1}{r}u_\theta \\ v_r & \frac{1}{r}v_\theta \end{pmatrix}}_{\text{polar } \underline{CR}: \begin{pmatrix} A & -B \\ B & A \end{pmatrix}} = \underbrace{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}_{\underline{CR}: \begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Lecture #11

Ex1: Determine domain of analyticity of $f(z) = \mathrm{Log}(5z-2i)$. Find $f'(z)$

► $5z - 2i \notin \mathbb{R}_{\leq 0} \iff z \notin \mathbb{R}_{\leq 0} + \frac{2}{5}i$

$$f'(z) = \frac{1}{5z-2i} \cdot 5 = \frac{1}{z - \frac{2i}{5}}$$

Ex2: a) Show that $\mathrm{Log}\sqrt{x^2+y^2}$ is a harmonic function on $\mathbb{R}^2 \setminus \{0\}$

b) Verify that $\mathrm{Arg}\, z$ is its harmonic conjugate on $D^* = \mathbb{R}^2 \setminus \{(x,0)\}_{x \in \mathbb{R}_{\leq 0}}$

c) Show that there is no harmonic conjugate on all $\mathbb{R}^2 \setminus \{(0,0)\}$.

► a) $\mathrm{Log}\sqrt{x^2+y^2} = \mathrm{Re}(\mathrm{Log}\, z) \Rightarrow$ harmonic on D^*

Likewise $\mathrm{Log}\sqrt{x^2+y^2} = \mathrm{Re}(\mathrm{Log}_0 z) \Rightarrow$ harmonic on $\mathbb{R}^2 \setminus \{(x,0) \mid x \in \mathbb{R}_{\geq 0}\}$ $\Big\} \Rightarrow$

$\Rightarrow$ it's harmonic on all $\mathbb{R}^2 \setminus \{(0,0)\}$.

b) $\mathrm{Arg}\, z = \mathrm{Im}(\mathrm{Log}\, z) \Rightarrow$ any harmonic conjugate on D^* is $\mathrm{Arg}\, z + C$ ← constant

c) It would be one of those in b), but $\mathrm{Arg}\, z$ is <u>discontinuous</u> on the ray $\mathbb{R}_{\leq 0} \Rightarrow$ no harmonic conjugate in $\mathbb{R}^2 \setminus \{(0,0)\}$

Ex3: Express inverse to cosine via logarithm.

► If $z = \cos^{-1}(w)$, then it's equivalent to $w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$. The latter is equivalent to $e^{2iz} - 2w \cdot e^{iz} + 1 = 0 \Rightarrow e^{iz} = \frac{2w \pm \sqrt{4w^2-4}}{2} = w \pm \sqrt{w^2-1}$ $\Big\} \Rightarrow$

(!) But: $\sqrt{\ldots}$ is already a multivalued function $\Rightarrow$ no need for $\pm$

$\Rightarrow e^{iz} = w + \sqrt{w^2-1} \iff iz = \log(w + \sqrt{w^2-1}) \iff \boxed{z = -i \log(w + \sqrt{w^2-1})}$

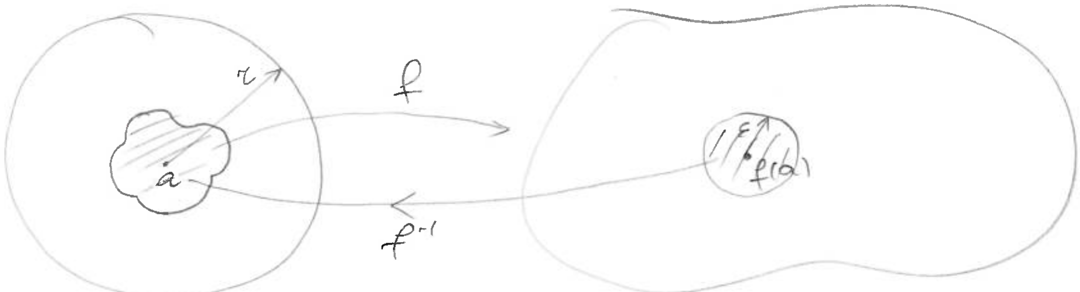
<u>composition</u> of <u>multivalued functions</u>

[<u>Note</u>: $\frac{d}{dw}\cos^{-1} w = -i \cdot \frac{1}{w + \sqrt{w^2-1}}\left(1 + \frac{w}{\sqrt{w^2-1}}\right) = \frac{-i}{\sqrt{w^2-1}} = -\frac{1}{\sqrt{1-w^2}}$ ← as in school!

Lecture #11

In fact our proofs of Theorems 1-2 can be naturally generalized:

Theorem 3 ("Complex inverse function theorem"): If f is analytic in $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is well-defined in small neighborhood $D_\varepsilon(f(a))$ and f^{-1} is analytic there with $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(f^{-1}(w))}$



▷ By the inverse function theorem in real analysis it suffices to check $\det \underbrace{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}_{\uparrow \text{"Jacobian of } f \text{ at point } a\text{"}} \neq 0$ where $f(x+iy) = u(x,y) + i \cdot v(x,y)$.

But CR $\Rightarrow u_x = v_y,\ v_x = -u_y \Rightarrow$ above $\det = u_x^2 + v_x^2 = |u_x + i v_x|^2 = \underbrace{|f'(a)|^2}_{\neq 0}$

Moreover, the Jacobian of the inverse map is given by $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1}$ which is again of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ ("rotation + magnification")

$\Rightarrow f^{-1}$ – analytic

Moreover, applying $\frac{d}{dw}$ to $w = f(f^{-1}(w))$, we get $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(f^{-1}(w))}$ ∎

§ 3.4 Washers, wedges, walls

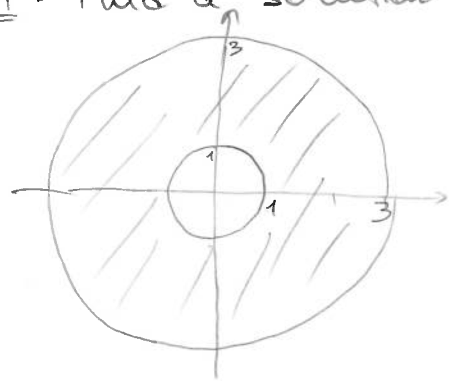
As we shall see latter in class, solving Laplace equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0$ in some domain in $\mathbb{C}$ can often be reduced to special domains:

"washer" "wedge" "walls"

Lecture #11

Let's show how one can find some of such solutions with boundary conditions
(! we are not finding all solutions but need just one now)

Ex 4: Find a solution $\phi(x,y)$ of Laplace equation in the domain



with initial condition:

$\phi|_{\text{inner circle}} = 10$

$\phi|_{\text{outer circle}} = 40$

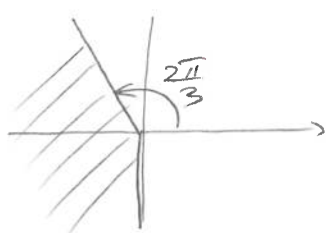
As ϕ is constant on these two circles it's logical to search for ϕ as a function of $\sqrt{x^2+y^2} = r$. But by Ex 2(a) know $\operatorname{Log}\sqrt{x^2+y^2}$ is harmonic.

Idea: For any constants A, B a function $A \cdot \operatorname{Log}\sqrt{x^2+y^2} + B$ is also harmonic!

Initial conditions:
$$\left.\begin{array}{l} A \cdot \underbrace{\operatorname{Log} 1}_{=0} + B = 10 \Rightarrow B = 10 \\ A \cdot \operatorname{Log} 3 + B = 40 \end{array}\right\} \Rightarrow A = \frac{30}{\ln 3}$$

So: $\boxed{\phi(x,y) = \frac{30}{\ln 3} \cdot \log\sqrt{x^2+y^2} + 10}$ is a solution

Ex 5: Find a solution $\phi(x,y)$ of Laplace's equation in the domain



with initial conditions

$\phi|_{\text{ray with angle } \frac{2\pi}{3}} = 10$

$\phi|_{\text{ray with angle } -\frac{\pi}{2}} = 60$

Idea: These two rays are level curves of function θ, hence its logical to look for the answer in the form $A \cdot \operatorname{Arg} z + B$

(!) Warning: $\operatorname{Arg} z$ is discontinuous on $\mathbb{R}_{\leq 0}$ inside our region though!

Lecture # 11

▶ A simple fix to the above issue is to take a different branch!

E.g. consider $A \cdot \underbrace{\arg_0 z}_{} + B$
harmonic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$

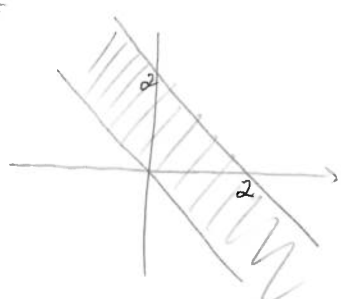
Find A, B from initial condition:

$$\begin{cases} A \cdot \frac{2\pi}{3} + B = 10 \\ A \cdot \frac{3\pi}{2} + B = 60 \end{cases} \xrightarrow{\text{solve}} \begin{cases} A = \frac{60}{\pi} \\ B = -30 \end{cases}$$

Not $-\frac{\pi}{2}$ b/c of our branch!

So: $\boxed{\frac{60}{\pi} \arg_0 z - 30}$ is a solution! ☐

Ex 6: Find a solution of Laplace eq-n in the depicted region with initial condition



$\phi|_{x+y=0} = 10$

$\phi|_{x+y=2} = 20$

▶ Clearly above lines are level curves of $x+y$. Any linear function is harmonic clearly! Hence, let's look for $A(x+y)+B$.

Initial condition: $\begin{cases} A \cdot 0 + B = 10 \\ A \cdot 2 + B = 20 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = 10 \end{cases}$

So: $\phi(x,y) = 5(x+y) + 10$ is a solution! ☐

Rmk: For washers and wedges centered at some $a \neq 0$ use the same idea by replacing $\mathrm{Log}|z| \rightsquigarrow \mathrm{Log}|z-a|$
$\arg_\tau(z) \rightsquigarrow \arg_\tau(z-a)$

relevant for homework!