Lecture #12 (§3.5–4.1)

Def: $F(z)$ is called a branch of a multivalued function $f(z)$ in a domain D if $F(z)$ is single-valued and continuous in D, and for any $z \in D$ the value $F(z)$ is one of the values of $f(z)$.

• **Ex1** Find a branch $F(z)$ of $f(z) = \log(z^2-4)$ which is analytic at $z=0$. Find $F(0)$, $F'(0)$.

▸ As $0^2-4=-4$, we shouldn't take $\mathrm{Log}(z^2-4)$! However, any other ray from the origin, not containing -4 would work!

For example, can take $F(z) = \mathrm{Log}_0(z^2-4)$ with $\arg_0 \in (0; 2\pi]$. Then

$F(0) = \ln 4 + i\cdot\pi$, $F'(0) \xlongequal{\text{chain rule}} \left.\frac{2z}{z^2-4}\right|_{z=0} = 0$

[**Note**: We could also take e.g. $F(z) = \mathrm{Log}_{4\pi}(z^2-4)$ with $\arg_{4\pi} \in (4\pi; 6\pi]$. In that case $F(0) = \ln 4 + i\cdot 5\pi$, $F'(0)=0$

• Last time we derived the formula for inverse cosine:

$$\boxed{\cos^{-1} w = -i \log(w + \sqrt{w^2-1})} \leftarrow \text{composition of two multivalued functions}$$

Note $\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z}$, hence, its principal branch is $e^{\frac{1}{2}\mathrm{Log} z}$.

Ex2: If $w \in (-1,1)$ and principal branches are used above, what is the range of $\cos^{-1} w$?

▸ $\sqrt{w^2-1} = e^{\frac{1}{2}\mathrm{Log}(w^2-1)} = e^{\frac{1}{2}(\ln(1-w^2)+i\pi)} = i\cdot\sqrt{1-w^2}$

But: $\sqrt{w^2 + \sqrt{1-w^2}^{\,2}} = 1$

$\Rightarrow \mathrm{Log}(w + i\sqrt{1-w^2}) = i\theta$ where $\theta \in (-\pi, \pi]$ s.t. $\begin{cases}\cos\theta = w \\ \sin\theta = \sqrt{1-w^2}\end{cases}$

This is precisely $\cos^{-1}(w)$ from school which ranges from 0 to π as w ranges from -1 to 1

Answer: $(0, \pi)$.

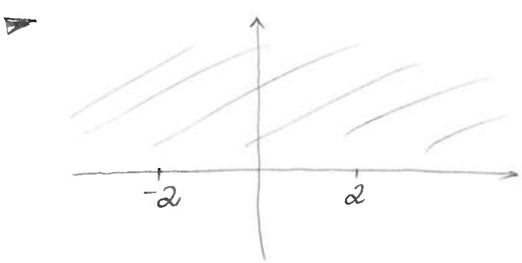
Lecture #12

Before starting a new topic, let's recall the last 3 Exercises from previous class - solving Laplace equation on washers, wedges and walls, with constant values on boundary components.

- at the moment we are not finding all solutions, but are just guessing one knowing that
  1) $\ln|z| = \ln\sqrt{x^2+y^2}$ is harmonic on $\mathbb{C}\setminus\{0\} = \mathbb{R}^2\setminus\{(0,0)\}$
  2) $\arg_\tau z$ is harmonic on $\mathbb{C}\setminus\{\text{ray with angle } \tau\}$
  3) $ax+by+c$ is harmonic on $\mathbb{C}$

Rmk: If your washer or wedge are centered not at the origin, but some point $z_0 = x_0+iy_0 \in \mathbb{C}$, then do parallel transport, i.e. look at $\ln\sqrt{(x-x_0)^2+(y-y_0)^2}$ and $\arg_\tau(z-z_0)$ for 1) & 2)

Ex3: Find a harmonic function $g(x,y)$ on upper half plane $\{z\in\mathbb{C} \mid \operatorname{Im} z>0\}$ i.e. $\{(x,y)\in\mathbb{R}^2 \mid y>0\}$ s.t. $\lim_{y\to 0^+} g(x,y) = \begin{cases} 0, & \text{if } x>2 \\ 1, & \text{if } -2<x<2 \\ 0, & \text{if } x<-2 \end{cases}$



Guess: look for $g(x,y)$ in the form

$$g(x,y) = A_1 \cdot \operatorname{Log}(z-2) + A_2 \operatorname{Log}(z+2) + B$$

(you could also use $\log_{-\pi/2}$ instead of Log!)

Then initial conditions give us: $\left.\begin{array}{l} B=0 \\ A_1\cdot\pi + B = 1 \\ A_1\cdot\pi + A_2\pi + B = 1 \end{array}\right\} \Rightarrow A_1 = 1/\pi,\ A_2 = -1/\pi,\ B=0$

Hence, a solution is given by $\frac{1}{\pi}(\operatorname{Arg}(z-2) - \operatorname{Arg}(z+2))$ ∎

Rmk: a) Here we were thinking of above region as combination of 2 wedges
b) Same argument would work if we had >2 points of discontinuity.

Lecture #12

§3.5 Complex powers

In analogy with $\sqrt[n]{z}$ from p. 1 as well as Lecture 2, we now define

Def: For any complex z, w (with $z \neq 0$), define the power
$$\boxed{z^w := e^{w \log z} = \{ e^{w(\mathrm{Log} z + i \cdot 2\pi k)} \mid k\text{-integer} \}}$$

Note: a) For w-integer, this coincides with Lecture 1 and has just 1 value

b) For $w = \frac{a}{b}$, $\gcd(a,b)=1$, a,b-integers, $b>0$ $\rightsquigarrow$ get b values

c) For w not rational, get infinitely many values.

Ex 4: Compute i^i.

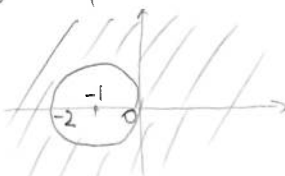
▸ $i^i = e^{i \log i} = e^{i(\ln 1 + i(\frac{\pi}{2} + 2\pi k))} = e^{-\frac{\pi}{2} - 2\pi k}$, k-integer ▯

Note: If the question was to find the principal value (i.e. value of the principal branch), then it would be $k=0$ case $e^{-\pi/2}$.

Ex 5: Find a branch of $(z^4-1)^{1/2}$ analytic in the exterior of the unit circle $|z|=1$

▸ Note that the image of the above domain D under $z \mapsto z^4 - 1$ is the dashed region

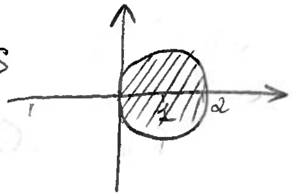
Ω



In particular, any ray starting from the origin contains points in Ω.

So: naive try $e^{\frac{1}{2}\log_\tau (z^4-1)}$ for some τ never works here!!!

Instead: Note that $(z^4-1)^{1/2} = z^2 \cdot (1 - z^{-4})^{1/2}$. Now the image of D under $z \xrightarrow{w=} 1 - z^{-4}$ is $|w-1|<1$ (interior of dashed disk)



Hence, e.g. can take $z^2 e^{\frac{1}{2}\mathrm{Log}(1-z^{-4})}$ ▯

Lecture #12

Next 2 weeks – complex integration

School : $\int_a^b f(x)dx$ (may have $a=-\infty$ and/or $b=+\infty$)

Now : $\int_\gamma f(z)dz$ ⟵ • f is a $\mathbb{C}$-valued function
• γ – some path in $\mathbb{C}$.

Clearly the key complication arises from γ!

Note : For γ = straight line segment on real axis from $a\in\mathbb{R}$ to $b\in\mathbb{R}$, the notion $\int_\gamma f(z)dz = \int_a^b f(z)dz$ should clearly reduce to school setup : $\int_a^b f(z)dz = \int_a^b u(x,0)dx + i\cdot\int_a^b v(x,0)dx$ with $f(x+iy) = u(x,y) + iv(x,y)$.

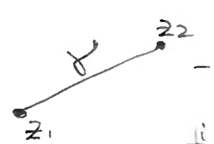
! Read at home Section 4.1 of the book, which discusses which γ shall appear and how we treat them.

Let me briefly summarize key concepts:

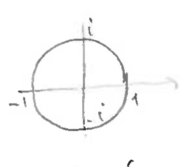
Def : A set γ in $\mathbb{C}$ is called a smooth arc if it is the range of a continuous $\mathbb{C}$-valued function $z : [a,b] \to \mathbb{C}$ such that
- $z'(t)$ is well-defined, continuous, nowhere vanishing on $[a,b]$
- $z(t)$ is a one-to-one map on $[a,b]$

Def : A set γ in $\mathbb{C}$ is called a smooth closed curve if it is the range of a continuous $\mathbb{C}$-valued function $z=z(t) : [a,b] \to \mathbb{C}$ s.t.
- $z'(t)$ is well-defined, continuous, nowhere vanishing on $[a,b]$
- $z(a)=z(b)$ AND $z(t)$ is one-to-one on $[a,b)$.

Examples :



– smooth arc , e.g. $z : [0,1] \to \mathbb{C}$
$t \mapsto z_1 + t(z_2 - z_1)$



– smooth closed curve, e.g. $z : [0, 2\pi] \to \mathbb{C}$
$t \mapsto e^{it}$

Note : See terminology "admissible" on p. 159 of book.

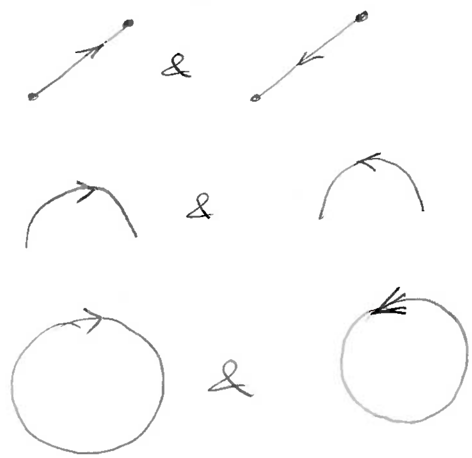
Lecture #12

Note that the choice of $z = z(t)$ in above two definitions is highly non-unique

Terminology: smooth curve γ in $\mathbb{C}$ will mean either of above two definitions.

Furthermore, any smooth curve admits two choices of orientations

Example:



Terminology: $\underbrace{\text{directed smooth arc, directed smooth closed curve}}_{\text{directed smooth curve}}$

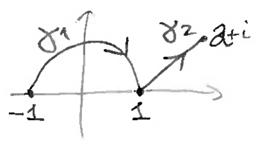
Finally, we shall be most generally working with the following:

Def: A contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that γ_{k+1} starts with the end-point of γ_k for all $k = 1, 2, \ldots, n-1$.

Notation: $\boxed{\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n}$

Note that Γ is directed itself! We use $-\Gamma$ to denote the contour with opposite orientation. If we forget orientation, then we recover just a piecewise smooth curve

Example:



$\Gamma = \gamma_1 + \gamma_2$,

$\gamma_1 : z_1(t) = \cos(\pi - t) + i\sin(\pi - t)$, $0 \le t \le \pi$

$\gamma_2 : z_2(t) = 1 + t(1+i)$, $0 \le t \le 1$

parametrization

rescale to parametrize by $[0,1]$.

of Γ by $[0, \pi+1]$

Lecture #12

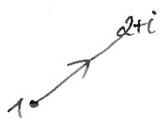
Note: if we parametrized $\gamma_1, \dots, \gamma_n$ by $[0,1]$, then to get a parametrization of $\Gamma = \gamma_1 + \dots + \gamma_n$ also by $[0,1]$ one can split $[0,1]$ into $[0, \frac{1}{n}] \cup [\frac{1}{n}, \frac{2}{n}] \cup \dots \cup [\frac{n-1}{n}, 1]$ and scale by n each segment, see e.g. Example 2 on pages 156–157.

Let's illustrate this by example above 

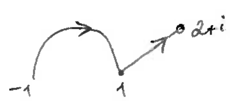
Instead of using $[0, \pi]$ we want to use $[0, \frac{1}{2}]$ $\Rightarrow$ replace t by $2\pi t$:

γ_1  $\gamma_1 : z_1(t) = \cos(\pi - 2\pi t) + i \sin(\pi - 2\pi t)$, $0 \le t \le \frac{1}{2}$

Instead of using $[0,1]$ we want to use $[\frac{1}{2}, 1]$ for γ_2:

γ_2  $\gamma_2 : z_2(t) = 1 + (2t - 1) \cdot (1+i)$, $\frac{1}{2} \le t \le 1$.

Combining the two we get:



$$\boxed{\Gamma : \; z(t) = \begin{cases} \cos(\pi - 2\pi t) + i \sin(\pi - 2\pi t), & 0 \le t \le \frac{1}{2} \\ 1 + (2t-1)(1+i), & \frac{1}{2} \le t \le 1 \end{cases}}$$

Terminology: Contour $\Gamma = \gamma_1 + \dots + \gamma_n$ is a *loop* (a.k.a. *closed contour*) if end-point of γ_n is the starting point of γ_1.

If there are no other multiple points then Γ is called *simple closed contour*.

Note that any simple closed contour determines interior & exterior parts of the complex plane, see Fig 4.11 in book.
interior → bounded; exterior → unbounded

Next time: Computation of $\int_\Gamma f(z)dz$ given some parametrization of Γ (clearly independent of it!)

Key: Often the result is independent of Γ and rather only depends on end/start points of Γ (and singularities of f in interior).