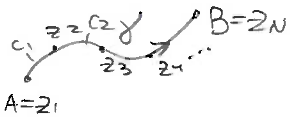
Lecture #13

Similarly to calculus one defines integrals formally as

$$\int_\gamma f(z)dz = \lim_{\substack{\text{max}\\ \text{length}\\ \text{b/w } z_i \,\&\, z_{i+1} \to 0}} \left\{ \sum_{i=1}^{N-1} f(c_i)\cdot(z_{i+1}-z_i) \right\}$$



While it's quite useless in practical computations, let's note some properties:

$$\boxed{\begin{gathered}\int_\gamma c\cdot f(z)dz = c\cdot\int_\gamma f(z)dz, \qquad \int_\gamma (f(z)+g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz \\ \int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz \quad \text{with } -\gamma \text{ being oppositely oriented path}\end{gathered}}$$

For practical reasons, we shall actually take another viewpoint. Note that $\underline{z_{i+1}-z_i \sim z'(t_i)\cdot(t_{i+1}-t_i)}$ where $z:[a,b]\longrightarrow\gamma$ - parametrization and $z_i = z(t_i)$

Therefore, for us:

$$\boxed{\int_\gamma f(z)dz = \int_a^b f(z(t))\cdot z'(t)dt}$$

← where $z:[a,b]\longrightarrow\gamma$ is some parametrization of smooth curve γ

<u>Note</u>: In particular, the result is actually independent of parametrization!

If Γ is a contour, i.e. $\Gamma = \gamma_1+\gamma_2+\ldots+\gamma_n$ with γ_i - smooth directed curves, then from above limit abstract definition it's clear that

$$\boxed{\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz}$$

In particular, if $\gamma = [a,b]\subseteq\mathbb{R}\subseteq\mathbb{C}$ then choosing $\begin{aligned} z: [a,b] &\to \gamma \\ t &\mapsto t\end{aligned}$

we get $\displaystyle\int_{[a,b]} f(z)dz = \int_a^b f(t)dt$ which is just $\displaystyle\int_a^b u(t,0)dt + i\cdot\int_a^b v(t,0)dt$

if $f(x+iy) = u(x,y)+i\cdot v(x,y)$

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Ex 1: a) Evaluate $\int_{\gamma'} e^{3z}dz$ with γ' being line segment from 1 to -1.

b) Evaluate $\int_{\gamma} e^{3z}dz$ with γ being an upper arch of unit circle from 1 to -1.

a) Know that $e^{3x} = \left(\frac{1}{3}e^{3x}\right)'$, hence by Fundamental Theorem of Calculus:

$$\int_{\gamma'} e^{3z}dz = \left(\frac{1}{3}e^{3x}\right)\Big|_{x=1}^{x=-1} = \frac{e^{-3}-e^{3}}{3}$$

b) If we were to use direct definition, then first we parametrize γ

E.g. $z: [0,\pi] \to \gamma$, $t \mapsto \cos t + i\sin t$. Then, our integral is just

$$\int_0^{\pi} e^{3(\cos t + i\sin t)} \cdot (-\sin t + i\cos t)dt = \int_0^{\pi} e^{3\cos t}\left(\cos(\sin t) + i\cdot\sin(\sin t)\right) \cdot (-\sin t + i\cos t)dt = \ldots ?$$

But good news we can still apply Fund. Thm. Calculus, since

$$e^{3z} = \left(\frac{1}{3}e^{3z}\right)' \text{ for all } z!$$

$$\Rightarrow \int_{\gamma} e^{3z}dz = \frac{e^{3z}}{3}\Big|_{z=1}^{z=-1} = \frac{e^{-3}-e^{3}}{3}$$

Fundamental Theorem of Calculus in present setup = [§ 4.3, Thm 6]:

If $f(z)$ is a $\mathbb{C}$-valued continuous function in a domain D and has antiderivative $F(z)$ on D, i.e. $F'(z) = f(z)$, then for any contour $\Gamma = \Gamma_A^B$ (A = starting point, B = end-point) lying in D, we have

$$\int_{\Gamma} f(z)dz = F(B) - F(A)$$

Corollaries: 1) $\int_{\Gamma} f(z)dz = 0$ for closed Γ in above setup.

2) $\int_{\Gamma_A^B} f(z)dz$ depends only on A, B not a curve, hence, we can replace Γ accordingly (like γ' instead of γ in Ex 1!)

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Proof of Thm

1) Assume first γ - smooth curve with parametrization $z : [a,b] \to \gamma$
$z(t) = x(t) + i\, y(t)$

Let's evaluate $\frac{d}{dt}(F(z(t)))$, where $F(x+iy) = u(x,y) + i\, v(x,y)$.

$$\frac{d}{dt}(F(z(t))) = \frac{d}{dt}\left(u(x(t), y(t)) + i \cdot v(x(t), y(t))\right) \stackrel{\text{usual chain rule}}{=}$$

$$= u_x \cdot x' + u_y \cdot y' + i\,(v_x \cdot x' + v_y \cdot y') \stackrel{CR}{=}$$

$$= u_x \cdot x' - v_x \cdot y' + i\,(v_x \cdot x' + u_x \cdot y') = \underbrace{(u_x + i v_x)}_{F'}(x' + iy')$$

$$\Downarrow$$

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t)$$

So: $\displaystyle\int_{\gamma_A^B} f(z)\,dz = \int_a^b \underbrace{f(z(t))}_{=F'(z(t))} \cdot z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt \stackrel{\text{usual Fund. Thm. Calculus}}{=} F(\underbrace{z(b)}_{=B}) - F(\underbrace{z(a)}_{=A})$

$$\Downarrow$$

$$\boxed{\int_{\gamma_A^B} f(z)\,dz = F(B) - F(A)}$$

2) Now if $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$, with γ_i from z_i to z_{i+1}, then:

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz = F(z_2) - F(z_1) + F(z_3) - F(z_2) + \ldots + F(z_{n+1}) - F(z_n)$$

$$\stackrel{\text{telescopic sum}}{=} F(\underset{\substack{\parallel \\ B}}{z_{n+1}}) - F(\underset{\substack{\parallel \\ A}}{z_1}) = F(B) - F(A).$$

$\blacksquare$

__Note__: The above result applies only when antiderivative exists!
If not, then just do direct calculation.

$$\boxed{\text{Fact: If } f(z) \text{ is continuous on } \gamma, \text{ then } \int_\gamma f(z)\,dz \text{ exists}}$$

(can you deduce it from a similar result in calculus?)

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The following is a mandatory exercise on the subject:

Ex2: Let $C_r(z_0)$ be a circle centered at z_0 of radius r oriented counter clockwise, and n be an integer. Compute

$$\int_{C_r(z_0)} (z-z_0)^n dz.$$

Proof 1

First parametrize $C_r(z_0)$ via e.g. $z(t) = z_0 + re^{it}$ with $0 \le t \le 2\pi$

Then $(z-z_0)^n = (re^{it})^n = r^n \cdot e^{int}$, $z'(t) = r \cdot i \cdot e^{it}$

$$\Rightarrow \int_0^{2\pi} r^n \cdot e^{int} \cdot ri \cdot e^{it} dt = r^{n+1} \cdot i \int_0^{2\pi} e^{i(n+1)t} dt$$

Case 1: $n \neq -1 \Rightarrow$ antiderivative of $e^{i(n+1)t}$ is $\frac{e^{i(n+1)t}}{i(n+1)}$

$$\Rightarrow \int_0^{2\pi} e^{i(n+1)t} dt = \frac{e^{i(n+1)t}}{i(n+1)} \Big|_{t=0}^{t=2\pi} = 0$$

Case 2: $n = -1 \Rightarrow$ get $i \int_0^{2\pi} 1 dt = 2\pi i$

Answer: $$\boxed{\int_{C_r(z_0)} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}}$$

Proof 2

- If $n \neq -1$, then $(z-z_0)^n = \left(\frac{1}{n+1}(z-z_0)^{n+1}\right)'$ which holds on all $\mathbb{C}$ with a possible exception of z_0 when $n < 0$

Hence by Fund. Thm. Calculus (see Corollary 1)): $\int_{C_r(z_0)} (z-z_0)^n dz = 0$

- If $n = -1$, then $(z-z_0)^{-1} = \frac{1}{z-z_0}$ does not have an antiderivative on any domain containing $C_r(z_0)$ since $\log(z-z_0)$ has no branch on $\mathbb{C} \setminus \{z_0\}$. However, if we split $C_r(z_0)$ into 2 archs you can still apply Fundam. Thm. Calculus – finish at home!