Lecture #14

Ex 1: Compute $\int_\Gamma f(z)dz$ where $f(z)=\frac{1}{z}$ and contour is:

a) Γ is a half-circle $-1 \quad 1$

b) Γ is a unit circle $-1 \quad \cdot \quad 1$ oriented counterclockwise

c) Γ is an ellipse $4x^2+y^2=1$ oriented counterclockwise.

a) Proof 1 (as last time): parametrize $z:[0,\pi]\to\Gamma$

$t \longmapsto \cos t + i\sin t = e^{it}$

Then $\int_\Gamma \frac{1}{z}dz = \int_0^\pi e^{-it}\cdot ie^{it}\,dt = i\pi$

Proof 2 (using Fundamental Theorem of Calculus = "F.T.C.")

$\mathrm{Log}_{-\frac{\pi}{2}} z$ is an antiderivative of $\frac{1}{z}$ on $\mathbb{C}\setminus\{\text{ray } \theta=-\pi/2\}$ which contains Γ

$\Rightarrow \int_\Gamma \frac{1}{z}dz = \mathrm{Log}_{-\pi/2} z\Big|_{z=1}^{z=-1} = (\ln 1 + i\arg_{-\pi/2}(-1)) - (\ln 1 + i\arg_{-\pi/2} 1) = \pi i$

b) We did the computation in a direct way last time to get $2\pi i$.

Also we noted the argument of Proof 2 from a) does not apply on the nose, as $\log z$ has no branch on open set containing $\underbrace{|z|=1}_{\text{unit circle}}$.

Fix 1: You can split the circle into two arches (e.g. top & bottom):

$$\int_{\circlearrowleft} \frac{1}{z}dz = \underbrace{\int_{-1 \frown 1} \frac{1}{z}dz}_{=\pi i \text{ by a)}} + \underbrace{\int_{-1 \smile 1} \frac{1}{z}dz}_{} = \pi i + \pi i = 2\pi i$$

can be computed as in a) but we need a branch of log that contains $-1 \smile 1$, e.g. $\mathrm{Log}_{\frac{\pi}{2}} z$.

$\Rightarrow \int_{-1 \smile 1} \frac{1}{z}dz = \mathrm{Log}_{\frac{\pi}{2}} z\Big|_{-1}^{1} = i\cdot 2\pi - i\cdot\pi = \pi i$

B & A approach $e^{i\cdot(-\pi/2)}$

Fix 2: $\int_{\circlearrowleft} \frac{1}{z}dz = \lim \int_{B\,A} \frac{1}{z}dz = \lim\left(\mathrm{Log}_{-\frac{\pi}{2}} z\Big|_A^B\right) = i\cdot\frac{3\pi}{2} - i\cdot\left(-\frac{\pi}{2}\right) = 2\pi i.$

c) Proof 1 fails, but either of Fix 1 or Fix 2 apply to give the same $2\pi i$

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As we shall see next week the above $\oint_{\Gamma} \frac{1}{z}dz$ is key to <u>all</u> course!

Next week, in the proof of several important results we would need bounds on integrals. To this end, recall that in calculus

$$\left|\int_a^b f(z)dz\right| \le M \cdot |b-a| \quad \text{if} \quad |f(z)| \le M \quad \text{for all } a \le z \le b$$

$$\boxed{\text{Lemma: } \left|\int_{\Gamma} f(z)dz\right| \le \max_{z\in\Gamma} |f(z)| \cdot \text{length}(\Gamma)}$$

▶ This follows immediately from above calculus result:

$$\left|\int_{\Gamma} f(z)dz\right| \le \int_a^b |f(z)| \cdot |z'(t)|dt \le \underbrace{\max_{z\in\Gamma} |f(z)|}_{\substack{\text{this Max exists} \\ \text{as } |f(z)|\text{-continuous}}} \cdot \underbrace{\int_a^b \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\,dt}_{ds}}_{\text{length}(\Gamma)}$$

parametrize $z: [a,b] \to \Gamma$
$z(t) = x(t) + i\cdot y(t)$

∎

Ex 2: Show that $\left|\int_C \frac{dz}{z^2 - i}\right| \le \frac{3\pi}{4}$ where C is a circle $|z|=3$ oriented counterclockwise

▶ $\text{length}(C) = 6\pi$.

As $|z|=3 \Rightarrow |z^2| = 9 \Rightarrow z^2 - i$ is a circle of radius 9 centered at $-i$

$\Rightarrow |z^2 - i| \ge 8$ (from triangle inequality or just draw a picture)

$\Rightarrow \frac{1}{|z^2-i|} \le \frac{1}{8}$.

Hence by Lemma, we get $\left|\int_C \frac{dz}{z^2-i}\right| \le 6\pi \cdot \frac{1}{8} = \frac{3\pi}{4}$

∎

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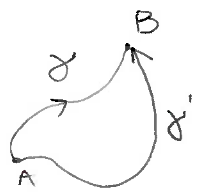
Theorem 1: Let f be a continuous function on a domain D. The following are equivalent:

1) f has an antiderivative in D
2) $\int_{\gamma_A^A} f(z)dz = 0$ for any closed contour $\gamma = \gamma_A^A$
3) $\int_{\gamma_A^B} f(z)dz = \int_{(\gamma')_A^B} f(z)dz$, i.e. path-independence. (γ_A^B, γ'^B_A – two paths in D starting at A, ending at B)

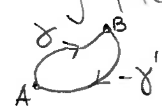
► 1) $\Rightarrow$ 2) by F.T.C.

2) $\Rightarrow$ 3):



equivalent to $\underbrace{\int_{\gamma_A^B} f dz - \int_{(\gamma')_A^B} f dz}_{\displaystyle \overset{\shortparallel}{\int f(z)dz}} = 0$

$\int f(z)dz = 0$ as $\gamma + (-\gamma')$ – closed.



3) $\Rightarrow$ 1): key is where to search for antiderivative ?!

But the guess is dictated by F.T.C: $\int_{\gamma_a^z} f(w)dw = F(z) - F(a)$ if F was an antiderivative of f. We shall use this as definition!

Fix any a in D, and consider $\boxed{F(z) := \int_{\gamma_a^z} f(w)dw}$

Note: [• property 3) guarantees the result is independent of γ_a^z
• different choice of a will change F only by adding constant

Remarks: verify that $\boxed{F'(z_0) = f(z_0) \text{ for any } z_0 \in D}$

Recall that $F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\int_{\gamma_{z_0}^z} f(w)dw}{z - z_0}$

the easiest way is to take $\gamma_a^z = \gamma_a^{z_0} + \gamma_{z_0}^z$

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(Continuation of the proof)

As f is continuous near $z_0 \Rightarrow f(z) = f(z_0) + g_1(z)$ and $g_1(z) \to 0$ as $z \to z_0$.

Thus: $\int_{\gamma_{z_0}^{z}} f(w)dw = f(z_0) \cdot \int_{\gamma_{z_0}^{z}} 1 dw + \int_{\gamma_{z_0}^{z}} g_1(w)dw = (z - z_0) f(z_0) + \int_{\gamma_{z_0}^{z}} g_1(w)dw.$

Hence: $$\frac{\int_{\gamma_{z_0}^{z}} f(w)dw}{z - z_0} - f(z_0) = \frac{\int_{\gamma_{z_0}^{z}} g_1(w)dw}{z - z_0}.$$

Since $g_1(w) \to 0$ as $w \to z_0$ we get $|g_1(w)| < \varepsilon$ ← any for w in small disk around z_0. But choosing $\gamma_{z_0}^{z}$ to be a straight line segment and using Lemma on p. 2, we get $\left| \frac{\int_{\gamma_{z_0}^{z}} g_1(w)dw}{z - z_0} \right| < \varepsilon$

This proves $F'(z_0) = f(z_0)$, as claimed. ∎

While the above result is cool, one may ask <u>how</u> do we check in practice if properties 2) or 3) hold. A partial answer is given by the following Key Result:

Theorem 2 ("Cauchy's Theorem"): Suppose $f(z)$ is analytic on a simple closed curve γ as well as inside it. Then $\int_\gamma f dz = 0$

Corollary: Thm 1 + Thm 2 $\Rightarrow$ analytic functions on interior of simple closed curve always have analytic antiderivative.

Warning: It's key that $f(z)$ is analytic not only on γ but also in the entire interior as example of $\int_{\circlearrowleft} \frac{1}{z} dz \neq 0$ shows.

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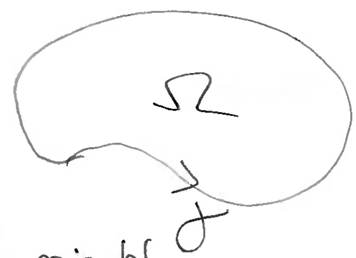
The proof of this theorem is based on the Green's Theorem:

$$\boxed{\int_\gamma P\,dx + Q\,dy = \iint_\Omega (Q_x - P_y)\,dA}$$



(here γ is a simple closed curve positively oriented
Ω = interior of γ, $dA = dx\,dy$
and we assume $Q_x = \frac{\partial Q}{\partial x}$, $P_y = \frac{\partial P}{\partial y}$ are continuous on Ω (and its open nbhd).)

<u>Recall</u>: If $z : [a,b] \to \gamma$ is a parametrization, then ($z(t) = (x(t), y(t))$)

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \left[P(x(t), y(t))\cdot x'(t) + Q(x(t), y(t))\cdot y'(t)\right]dt$$

<u>Proof of Theorem 2</u>

$f(x+iy) = u(x,y) + i\cdot v(x,y)$

<u>Note</u>: f-analytic $\Rightarrow$ u, v - C^1-smooth $\Rightarrow$ Green thm applies.

<u>So</u>:

$$\int_\gamma f(z)\,dz = \int_a^b \big(u(x(t), y(t)) + i\cdot v(x(t), y(t))\big)\cdot\big(x'(t) + i y'(t)\big)\,dt$$

$$= \int_a^b (u\cdot x' - v\cdot y')\,dt + i\int_a^b (v x' + u y')\,dt$$

$$= \int_\gamma (u\,dx - v\,dy) + i\cdot\int_\gamma (v\,dx + u\,dy) \quad \underline{\underline{\text{Green's Theorem}}}$$

$$= \iint_\Omega (-v_x - u_y)\,dA + i\iint_\Omega (u_x - v_y)\,dA$$

But Cauchy-Riemann eqs $\Rightarrow$ $u_x - v_y = 0 = -v_x - u_y$

<u>Hence</u>: $\int_\gamma f\,dz = 0$ as claimed. ∎

<u>Notations</u>: Given a closed curve Γ in $\mathbb{C}$-plane, its common to use
$\oint_\Gamma f(z)\,dz$ for positive orientation of Γ and $\ointclockwise_\Gamma f(z)\,dz$ for negative orientat.

<u>Next time</u>: Section 4.5.