Lecture #15 (§ 4.4 – 4.5)

* Start with presenting the proof of Cauchy Theorem = Theorem 2 of Lecture 14

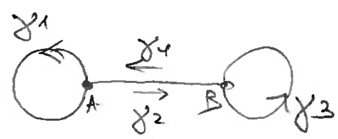
Question: Does the result remain true if replacing "simple closed curve" with "closed contour"?

Let's illustrate on some examples:

- if $\Gamma =$  , then we can break Γ into two smooth curves $\Gamma = \gamma_1 + \gamma_2$

$$\Rightarrow \int_\Gamma f(z)dz = \underbrace{\int_{\gamma_1} f(z)dz}_{=0} + \underbrace{\int_{\gamma_2} f(z)dz}_{=0}$$

by Cauchy Thm

- if $\Gamma =$  with γ_2, γ_4 being the same line segment AB but oriented different ways

$$\Rightarrow \int_\Gamma = \underbrace{\int_{\gamma_1}}_{=0 \text{ by Cauchy Theorem}} + \int_{\gamma_2} + \underbrace{\int_{\gamma_3}}_{=0\ -\text{''}-} + \int_{\gamma_4}$$

with $f(z)dz$ in all integrals

but $\int_{\gamma_2} = -\int_{\gamma_4} \Rightarrow \int_{\gamma_2} + \int_{\gamma_4} = 0.$

This can be summarized by:

Theorem 1: Suppose Ω is a domain without holes and let f be analytic in Ω. Then $\int_\Gamma f dz = 0$ for any closed contour $\Gamma \subset \Omega$

Corollary: Analytic functions in domains without holes have antiderivative

! **Warning**: No true in arbitrary domains, see e.g. $f(z) = \frac{1}{z}$ in $\Omega = \{1 < |z| < 2\}$

Lecture #15

Ex2: Let Γ be a simple closed contour (on the complex plane) positively oriented and $a \notin \Gamma$. Find $\int_\Gamma \frac{1}{z-a}\,dz$.

Case 1: $a \in$ outside of Γ.

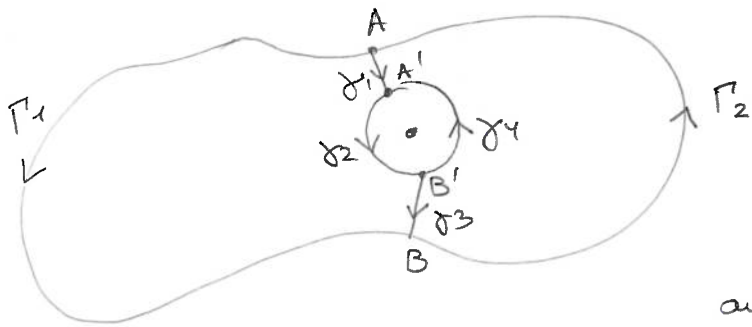
In this case $\frac{1}{z-a}$ is analytic on Γ & inside $\overset{\text{Thm 1}}{\Longrightarrow} \int_\Gamma \frac{1}{z-a}\,dz = 0$.

Case 2: $a \in$ inside of Γ.

We know that if γ is a circle (of any radius), centered at a and positively oriented then $\int_\gamma \frac{1}{z-a}\,dz = 2\pi i$.

$$\boxed{\text{Claim: } \int_\Gamma \frac{1}{z-a}\,dz = \int_\gamma \frac{1}{z-a}\,dz}$$

Let's illustrate it by picture-type argument:



Pick points A, A', B, B' as on picture. Let γ_1 be an oriented line segment from A to A', γ_3: from B' to B, and γ_2, γ_4 – two arches of circle γ, while Γ_1, Γ_2 – two parts of Γ.

Then: $\Gamma_1 + (-\gamma_3) + (-\gamma_2) + (-\gamma_1)$ is a closed curve and $\frac{1}{z-a}$ is analytic on it & inside, hence:

$$\boxed{\int_{\Gamma_1} \frac{dz}{z-a} = \int_{\gamma_1+\gamma_2+\gamma_3} \frac{dz}{z-a}}$$

Likewise, $\frac{1}{z-a}$ is analytic on & inside of $\Gamma_2 + \gamma_1 + (-\gamma_4) + \gamma_3 \Rightarrow$

$$\rightarrow \boxed{\int_{\Gamma_2} \frac{dz}{z-a} = \int_{(-\gamma_3)+\gamma_4+(-\gamma_1)} \frac{dz}{z-a}}$$

Thus: $$\int_\Gamma \frac{dz}{z-a} = \int_{\Gamma_1+\Gamma_2} \frac{dz}{z-a} = \underbrace{\left(\int_{\gamma_1} \frac{dz}{z-a} + \int_{-\gamma_1} \frac{dz}{z-a}\right)}_{=0} + \underbrace{\left(\int_{\gamma_3} \frac{dz}{z-a} + \int_{-\gamma_3} \frac{dz}{z-a}\right)}_{=0} + \overbrace{\int_{\gamma_2+\gamma_4=\gamma} \frac{dz}{z-a}}^{=2\pi i} = \underline{2\pi i}$$

Lecture #15

We can summarize the previous exercise as:

$$\boxed{\int_{\Gamma} \frac{dz}{z-a} = \begin{cases} 0 & \text{if } a \text{ is outside } \Gamma \\ 2\pi i & \text{if } a \text{ is inside } \Gamma \end{cases}}$$

Γ positively oriented simple closed contour

This formula is key to many other calculations!

Ex 3: Evaluate $\displaystyle\int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz$ where Γ is an ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ oriented counterclockwise

Step 1: Partial fraction decomposition

$$\frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} \Longrightarrow \begin{cases} A = 2/3 \\ B = 1/3 \end{cases}$$

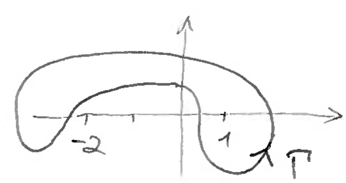
$$\Rightarrow \int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz = \frac{2}{3}\int_{\Gamma}\frac{dz}{z+2} + \frac{1}{3}\int_{\Gamma}\frac{dz}{z-1}$$

Step 2: Apply Ex 2 to each term.

Note that both $1, -2$ lie inside of $\Gamma \Rightarrow \displaystyle\int_{\Gamma}\frac{dz}{z+2} = 2\pi i = \int_{\Gamma}\frac{dz}{z-1}$

So: We get $\frac{2}{3}\cdot 2\pi i + \frac{1}{3}\cdot 2\pi i = \boxed{2\pi i}$

Ex 4: Evaluate $\displaystyle\int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz$ with Γ being



By previous exercise: $\displaystyle\frac{z}{(z+2)(z-1)} = \frac{2}{3}\cdot\frac{1}{z+2} + \frac{1}{3}\cdot\frac{1}{z-1}$

However, while 1 is inside Γ, -2 is actually outside!

$$\Rightarrow \int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz = \frac{2}{3}\cdot 0 + \frac{1}{3}\cdot 2\pi i = \boxed{\frac{2\pi i}{3}}$$

This strategy allows to integrate rational functions over any closed Γ. But how about more complicated functions, e.g. $\frac{e^{z^2}}{z-1}$?

Lecture #15 §4.5

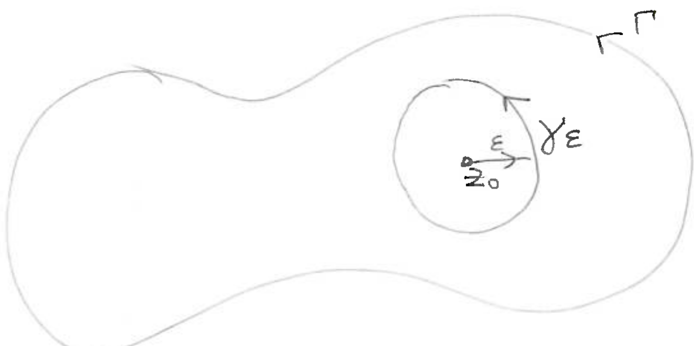
Theorem 2: Let Γ be a simple closed contour positively oriented. If f is analytic on Γ & inside it, and z_0 is any point inside Γ

$$\Rightarrow \boxed{\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)} \leftarrow \text{"Cauchy's Integral Formula"}$$

▶ Proof is quite elementary and is similar to our previous arguments

Step 1: As in Ex 2, we see that $\oint_\Gamma \frac{f(z)}{z-z_0}dz = \oint_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}dz$ for any $\varepsilon > 0$.

Here γ_ε denotes a radius ε circle centered at z_0, such that it lies inside Γ.



Know: $\int_{\gamma_\varepsilon} \frac{dz}{z-z_0} = 2\pi i \Rightarrow f(z_0) = \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(z_0)}{z-z_0}dz$

Step 2: Prove $\int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}dz \xrightarrow[\varepsilon\to 0]{} \int_{\gamma_\varepsilon} \frac{f(z_0)}{z-z_0}dz$.

Indeed, using estimates from last time, we get:

$$\left|\int_{\gamma_\varepsilon} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \operatorname*{Max}_{z\in\gamma_\varepsilon}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \cdot \text{length}(\gamma_\varepsilon) =$$

$$= \operatorname*{Max}_{z\in\gamma_\varepsilon} |f(z)-f(z_0)| \cdot \frac{1}{\varepsilon}\cdot 2\pi\varepsilon$$

But f - analytic $\Rightarrow f$ - continuous at $z_0 \Rightarrow \operatorname*{Max}_{z\in\gamma_\varepsilon}|f(z)-f(z_0)| \xrightarrow{\varepsilon\to 0} 0$

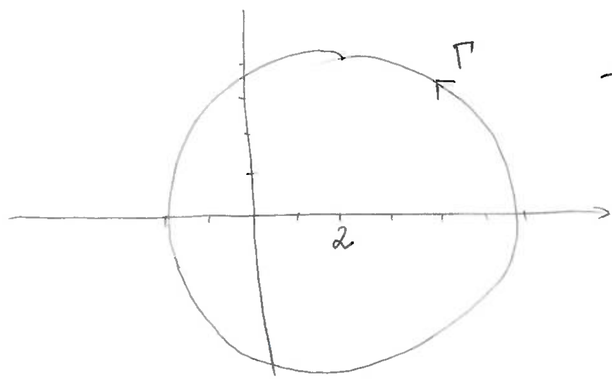
This completes the proof!

Lecture #15

Surprising consequence:

Fact: If Γ is a simple closed curve and $f(z)$ is analytic on Γ and inside it, then values of f on Γ determine all values inside Γ!

Ex 5: $\int_\Gamma \frac{e^{z^2}}{z-1}\,dz = ?$, where Γ is a positively oriented circle $|z-2|=4$.



$f(z) = e^{z^2}$ is an entire function
1 is inside Γ
$\Downarrow$

$$\int_\Gamma \frac{e^{z^2}}{z-1}\,dz = 2\pi i \cdot f(1) = \underline{\underline{2\pi i \cdot e}}$$

Next time: finish § 4.5, start § 4.6.