

Lecture #16 (§4.5)

Ex 1: Evaluate $\int_{\Gamma} \frac{e^{z^2}}{z^2-4} dz$ when:

- a) Γ is the unit circle $|z|=1$ oriented counterclockwise.
- b) Γ is the circle $|z|=3$ positively oriented.
- c) Γ is the circle $|z-1|=2$ positively oriented.

a) $\frac{e^{z^2}}{z^2-4}$ is analytic on unit disk $|z| \leq 1 \Rightarrow \int_{\Gamma} \frac{e^{z^2}}{z^2-4} dz = 0$

b) As $z^2-4 = (z-2)(z+2)$ and both ± 2 are inside Γ we need to be more careful now and shall rather use Cauchy's Integral Formula

Partial fraction decomposition: $\frac{1}{z^2-4} = \frac{1}{4} \cdot \frac{1}{z-2} - \frac{1}{4} \cdot \frac{1}{z+2}$

$$\Rightarrow \underbrace{\frac{1}{4} \int_{\Gamma} \frac{e^{z^2}}{z-2} dz}_{= 2\pi i \cdot e^{z^2} \Big|_{z=2}} - \underbrace{\frac{1}{4} \int_{\Gamma} \frac{e^{z^2}}{z+2} dz}_{= 2\pi i \cdot e^{z^2} \Big|_{z=-2}} = \frac{2\pi i}{4} (e^4 - e^4) = 0.$$

c) Only 2 is contained inside Γ (while -2 is outside!)

$$\Rightarrow \frac{1}{4} \int_{\Gamma} \frac{e^{z^2}}{z-2} dz - \frac{1}{4} \int_{\Gamma} \frac{e^{z^2}}{z+2} dz = \frac{2\pi i e^4}{4} = \frac{\pi i e^4}{2}.$$

Note: In part c) one could also argue by rather viewing $\frac{e^{z^2}}{z^2-4}$ as $\frac{e^{z^2}/(z+2)}{z-2}$ and as $\frac{e^{z^2}}{z+2}$ is analytic inside $\Gamma \Rightarrow$ get $2\pi i \cdot \left(\frac{e^{z^2}}{z+2} \right) \Big|_{z=2} = \frac{\pi i e^4}{2}$.
 Similar logic can also be applied in b).

Def: A domain D is called simply-connected (a.k.a. "without holes") if any simple closed contour lying in D has interior wholly in D

Ex 2: Which of the following are simply-connected:



Lecture #16

Main result last time was the Cauchy's Integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w-z} dw$$

- Γ is any positively oriented simple closed curve
- f is analytic inside & on Γ
- z is inside Γ .

As the right-hand side only involves values of f on Γ , we get a quite surprising result (with assumptions as above):

Corollary: Values of f on Γ determine values of f inside Γ !

Looking at the top boxed formula, and assuming derivative w.r.t. z and integration w.r.t. dw can be swapped, we arrive at "guess":

$$f'(z) = \frac{1}{2\pi i} \int_{\Gamma} \left[\frac{d}{dz} \left(\frac{f(w)}{w-z} \right) \right] dw = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^2} dw$$

We shall now prove this rigorously in a bigger generality:

Theorem 1: Let g be a continuous function on a contour Γ . For each $z \notin \Gamma$, consider $G(z) := \int_{\Gamma} \frac{g(w)}{w-z} dw$ (well-defined as $\frac{g(w)}{w-z}$ is continuous!)

Then G is analytic at each $z \notin \Gamma$ and

$$G'(z) = \int_{\Gamma} \frac{g(w)}{(w-z)^2} dw$$

When Γ is a positively oriented simple closed curve we get above

$$f'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^2} dw \quad \text{for any } z \text{ inside } \Gamma$$

By definition, we have

$$G'(z) = \lim_{\Delta z \rightarrow 0} \frac{G(z+\Delta z) - G(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \left\{ \frac{1}{\Delta z} \int_{\Gamma} \left(\frac{g(w)}{w-z-\Delta z} - \frac{g(w)}{w-z} \right) dw \right\}$$

$$= \lim_{\Delta z \rightarrow 0} \int_{\Gamma} \frac{g(w)}{(w-z)(w-z-\Delta z)} dw$$

Want: $\int_{\Gamma} \frac{g(w)}{(w-z)(w-z-\Delta z)} dw \xrightarrow{\Delta z \rightarrow 0} \int_{\Gamma} \frac{g(w)}{(w-z)^2} dw$

Lecture #16

(continuation of proof)

$$\int_{\Gamma} \frac{g(w)}{(w-z)(w-z-\Delta z)} dw - \int_{\Gamma} \frac{g(w)}{(w-z)^2} dw = \int_{\Gamma} \frac{g(w) \cdot \Delta z}{(w-z)^2 (w-z-\Delta z)} dw$$

Remains: $\int_{\Gamma} \frac{g(w) \cdot \Delta z}{(w-z)^2 (w-z-\Delta z)} dw \xrightarrow{\Delta z \rightarrow 0} 0$

But as $z \notin \Gamma$, there is some $\rho \in \mathbb{R}_{>0}$ such that $|w-z| \geq \rho$ for any $w \in \Gamma$.
 Hence, if $|\Delta z| < \rho/2$, we get $\left| \frac{1}{(w-z)^2 (w-z-\Delta z)} \right| \leq \frac{1}{\rho^2 \cdot \rho/2} = \frac{2}{\rho^3}$.

Also, as g is continuous on Γ , it's bounded: $|g(w)| \leq M$ for all $w \in \Gamma$ for some real M .

So: $\left| \int_{\Gamma} \frac{g(w) \Delta z}{(w-z)^2 (w-z-\Delta z)} dw \right| \leq |\Delta z| \cdot \underbrace{M \cdot \frac{2}{\rho^3} \cdot \text{length}(\Gamma)}_{\text{constant}} \rightarrow 0 \text{ as } |\Delta z| \rightarrow 0$

Same argument can be also applied to prove:

Theorem 1': Let g be a continuous function on a contour Γ . For each $z \notin \Gamma$ consider $G(z) := \int_{\Gamma} \frac{g(w)}{(w-z)^n} dw$. Then G is analytic at each $z \notin \Gamma$ and

$$G'(z) = n \cdot \int_{\Gamma} \frac{g(w)}{(w-z)^{n+1}} dw$$

In the particular case when Γ is a positively oriented simple closed contour and z is inside Γ , we get:

Theorem 2 ("higher order Cauchy theorem") : If f is analytic on & inside Γ , then

$$f^{(n)}(z) = \frac{1}{2\pi i} \cdot n! \cdot \oint_{\Gamma} \frac{f(w)}{(w-z)^{n+1}} dw$$

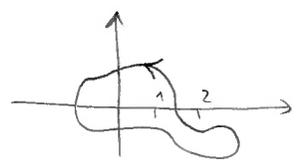
for any z inside Γ , where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$

Lecture #16

We shall often apply this result written in the form:

$$\int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}} dw = \frac{2\pi i}{n!} f^{(n)}(z)$$

Ex 3: Evaluate $\int_{\Gamma} \frac{e^{z^2}}{z^2(z-2)} dz$ with contour Γ on



Note that while 0 is inside Γ , 2 is outside!

Write $\frac{e^{z^2}}{z^2(z-2)} = \frac{e^{z^2}/(z-2)}{z^2}$, and apply above formula with $n=1$

and $f(z) = \frac{e^{z^2}}{z-2}$ which is analytic on & inside Γ to get:

$$\int_{\Gamma} \frac{e^{z^2}}{z^2(z-2)} dz = 2\pi i \cdot f'(0) = 2\pi i \cdot \left. \frac{e^{z^2} \cdot 2z(z-2) - e^{z^2} \cdot 1}{(z-2)^2} \right|_{z=0} = -\frac{\pi i}{2}$$

Let's summarize several corollaries of previous theorems.

Theorem 3: If f is analytic in a domain D , then all its derivatives $f^{(n)}$ exist and are analytic in D ! (D -any domain here, in particular, it does not have to be simply connected)

↑ Note: this is very different from usual calculus case!

Theorem 4: If $f = u + iv$ is analytic in a domain D , then all partials of u, v exist & continuous in D (i.e. $u, v - C^\infty$ -smooth)

Finally, in combination with Lecture 14, we get:

Theorem 5 ("Morera's Theorem") : If f is continuous in a domain D and $\oint_{\Gamma} f(z) dz = 0$ for every closed contour Γ in D , then f is analytic!

By Lecture 14, above implies that f has antiderivative F , i.e. $F' = f$. Thus F is analytic, hence so is $f = F'$ by Theorem 3!