Lecture #18

- Last time:

1) Liouville's theorem: any bounded entire function is constant.

$\Downarrow$

2) Fundamental theorem of algebra: any non-constant polynomial has roots $/\mathbb{C}$

(key idea: given a nowhere vanishing polynomial $p(z)$, apply Liouville's thm to $1/p(z)$)

3) Mean-value property (assuming f is analytic on $D_R(z_0)$)

$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{it})\,dt}$$

$\Downarrow$

4) An analytic function f on any domain D and its boundary, s.t. $|f(z)|$ achieves max in a (interior) point of D is constant.

$\Downarrow$

5) Maximum modulus principle.

Ex1: Prove that if $f(z)$ - entire function with $\operatorname{Re} f(z) \le M$ for all z, then $f \equiv$ constant.

► Consider $g(z) = e^{f(z)}$. Then $g(z)$ - entire, and bounded $|g(z)| = e^{\operatorname{Re} f(z)} \le e^M$.
Hence by Liouville's theorem, $e^{f(z)} \equiv$ constant $\underset{f\text{-continuous}}{\Longrightarrow}$ $f(z)$ - constant ■

Ex2: Find $\max\limits_{|z|\le 1} |az^n+b|$.

► By triangular inequality $|az^n+b| \le |a|\cdot|z|^n + |b| \le |a|+|b|$.
The above bound $|a|+|b|$ is indeed max: take z on the unit circle so that $az^n \in \mathbb{R}_{>0}\cdot b$, e.g. take $z=e^{i\theta}$ with $n\theta = \operatorname{Arg} b - \operatorname{Arg} a$
(if a or b are zero, the answer is obvious) ■

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We shall now generalize all results from Lecture 17 to harmonic f-s.
First, let's give now a short self-contained proof of [Lecture 8, Thm 2]

Theorem 1: Let $u = u(x,y)$ be a harmonic function on a simply connected domain D. Then there is an analytic $f(z)$ on D with $\mathrm{Re}\, f = u$

↑ in other words, u has a harmonic conjugate.

▸ If $f = u + iv$ is such an analytic f-n, then $f' = \begin{cases} u_x + i \cdot v_x \\ v_y - i \cdot u_y \end{cases}$ and evoking CR equations we can write $f' = u_x - iu_y$. This gives us the key idea:

Step 1: Consider $g(z = x+iy) = u_x - iu_y$. Then g is analytic

Indeed, we just need to check CR:

- $u_{xx} = (-u_y)_y$ follows from u being harmonic
- $(u_x)_y = -(-u_y)_x$ follows from equality of mixed partials.

Step 2: $g(z)$ has an antiderivative $F(z)$ on a simply connected D.

Write $F(z) = U + iV$. Comparing $F'(z) = U_x - iU_y$ to $g(z) = u_x - iu_y$,

we get $U_x = u_x$, $U_y = u_y$ $\Rightarrow$ $U - u = c$ - constant.

Step 3: Function $f(z) := F(z) - c$ is the desired one. ∎

Theorem 2 (Liouville's theorem for harmonic functions): If $u(x,y)$ is harmonic on $\mathbb{R}^2$ and bounded $u(x,y) \le M$ for all x,y, then $u \equiv \text{const}$.

▸ As $\mathbb{C} = \mathbb{R}^2$ is simply connected, by Thm 1 we have an entire f-n $f = u + iv$. Then $u = \mathrm{Re}\, f$ and we can apply Ex 1 to conclude that $f \equiv \text{const}$ $\Rightarrow$ $u \equiv$ constant. ∎

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Lemma (Mean-value property for harmonic): If u is harmonic on $D_R(z_0)$ then $\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{it})\,dt}$

▶ Applying Thm 1, pick an analytic $f(z) = u(z) + iv(z)$ on $D_R(z_0)$.
By mean-value property for analytic functions, we have
$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt$. Taking real parts of both sides implies the claim ∎

This result has a number of consequences (as last time):

Corollary 1 ("Max principle"): A harmonic function on domain D and its boundary attains its maximum on the boundary

Note that if u - harmonic $\Rightarrow$ $-u$ - harmonic, and $\max(-u) = -\min u$.
Hence, in contrast to analytic setup, we have min principle:

Corollary 2 ("Min principle"): A harmonic function on domain D and its boundary attains its minimum on the boundary.

Likewise, we have:

Corollary 3: If u is harmonic in a domain D and its boundary and achieves its max or min in a (interior) point of D, then $u \equiv$ const.

Combining these results, we obtain:

Theorem 3: Let $u_1(x,y), u_2(x,y)$ be harmonic functions in a bounded domain D and its boundary, such that $u_1 = u_2$ on the boundary of D. Then $u_1 = u_2$ on all D!

▶ Apply Cor 1 & 2 to harmonic $u_1 - u_2$ which is 0 on boundary of D ∎

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Ex 3: Find all solutions $\phi(x,y)$ of Laplace equation in the washer $1 \le |z| \le 3$ with initial conditions $\phi|_{|z|=1} = 10$, $\phi|_{|z|=3} = 40$.

▸ In [Lecture 11, Ex 4] we found one such solution:

$$\phi(x,y) = \frac{30}{\ln 3} \log\sqrt{x^2+y^2} + 10.$$

As washer is bounded, above thm implies this solution is unique! ■

Ex 4: Is theorem 3 true for unbounded domains?

▸ No. Let $D = \{(x,y) \mid y \ge 0\}$ - upper half plane. Then $y, 2y$ are both harmonic and both vanish on the boundary of D, i.e. $\{(x,0) \mid x \in \mathbb{R}\}$ ■

In general, solving a Laplace equation $\nabla^2\phi = 0$ on D with initial condition $\phi|_{\partial D = \text{boundary of } D}$ given is called Dirichlet Problem

$\boxed{\text{Q: Can one write explicit solution of Dirichlet Problem analogously to Cauchy integral formula?}}$

Recall: If f is analytic in $D_R(0)$, then for any $|z| < R$:

$$\boxed{2\pi i \cdot f(z) = \int_{C_R(0)} \frac{f(w)}{w-z}\,dw}$$

! However, in contrast to the mean-value property, taking real parts is not immediate.

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Trick: Use $\displaystyle\int_{C_R(0)} \frac{f(w)\bar{z}}{R^2 - w\bar{z}}\,dw = 0$ as $\displaystyle\frac{f(w)\cdot\bar{z}}{R^2 - w\bar{z}}$ is an analytic function of w on & inside $C_R(0)$.

$$\Downarrow$$

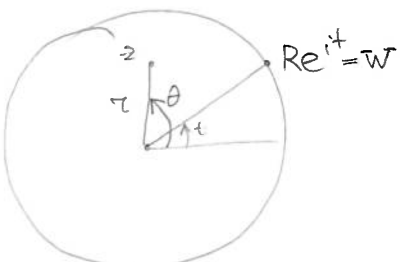
$$f(z) = \frac{1}{2\pi i}\int_{C_R(0)}\left(\frac{f(w)}{w-z} + \frac{f(w)\bar{z}}{R^2 - w\bar{z}}\right)dw = \frac{1}{2\pi i}\int_{C_R(0)} \frac{R^2 - z\cdot\bar{z}}{(w-z)(R^2 - w\bar{z})} f(w)\,dw$$

Let's compute the latter integral directly by parametrizing $C_R(0)$ via $w = Re^{it}$, $0 \le t \le 2\pi$:

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - Re^{it}\bar{z})} f(Re^{it})\cdot iRe^{it}\,dt$$

$$= \frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{(Re^{it} - z)(Re^{-it} - \bar{z})}\,dt = \boxed{\frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2}\,dt}$$

here denominator is real!



Finally if $z = re^{i\theta}$, then by the law of cosines:

$$\boxed{|Re^{it} - z|^2 = R^2 + r^2 - 2rR\cos(\theta - t)}$$

This implies the following fundamental property (by taking Re)

Theorem 4 (Poisson integral formula): If ϕ is harmonic in $D_R(0)$, then

$$\phi(re^{i\theta}) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{\phi(Re^{it})}{R^2 + r^2 - 2rR\cos(\theta - t)}\,dt$$

this is an explicit solution of the Dirichlet Problem on a disk

(the function $\displaystyle\frac{R^2 - r^2}{2\pi(R^2 + r^2 - 2rR\cos(\theta - t))}$ is called Poisson kernel)

Next time: Review session for Exam 1