Lecture #23

Last time:

1) Radius of convergence is given by Hadamard formula
2) $f(z)$ - analytic in $D_R(z_0) \Rightarrow$ its Taylor series converges in $D_R(z_0)$
3) if $\sum a_n (z-z_0)^n$ has R.O.C. $R>0$, then it converges to analytic function $f(z)$ in $D_R(z_0)$

Let's conclude this with the following:

Theorem 1: If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has R.O.C. $R>0$ and $f(z)$ is its analytic value inside $D_R(z_0)$, then $a_n = \frac{f^{(n)}(z_0)}{n!}$, so that the Taylor series of $f(z)$ at z_0 is precisely above $\sum a_n (z-z_0)^n$

Let $C = \underbrace{C_r(z_0)}_{\text{circle of radius } r \text{ centered at } z_0}$ for any $0<r<R$.

By Higher Order Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

But $\sum_{k=0}^{\infty} a_k (z-z_0)^k \xrightarrow[\text{on } D_r(z_0)]{\text{uniformly}} f(z)$

$\xRightarrow{\text{Lemma 3 of last lecture}}$ $f^{(n)}(z_0) = \frac{n!}{2\pi i} \sum_{k=0}^{\infty} \int_C a_k (z-z_0)^{k-n-1} dz$

Recall: $\oint_C (z-z_0)^s dz = \begin{cases} 2\pi i & \text{if } s=-1 \\ 0 & \text{if } s \neq -1 \end{cases}$

Hence: $f^{(n)}(z_0) = \frac{n!}{2\pi i} \cdot a_n \cdot 2\pi i = n! \cdot a_n \Rightarrow \boxed{a_n = \frac{f^{(n)}(z_0)}{n!}}$

Upshot: Convergence disk $D_R(z_0)$ with $R = $ R.O.C. and Taylor disk, i.e. max disk centered at z_0 on which $f(z)$ - analytic do coincide!

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Rule of thumb (for problems on R.O.C):

1) try ratio test
2) try root test or Hadamard formula
3) try to relate to Taylor disk using above "Upshot".

It will be also useful to record the following 2 results:

Proposition 1: If $f(z)$ - analytic at z_0, then Taylor series for $f'(z)$ at z_0 is obtained by termwise differentiation of Taylor series of $f(z)$, and converges in the same disk

Example: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \Rightarrow \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

Proposition 2: If $f(z) = \sum_{n\geq 0} a_n (z-z_0)^n$, $g(z) = \sum_{n\geq 0} b_n (z-z_0)^n$ are Taylor series, then

$$\left.\begin{aligned} (f+g)(z) &= \sum_{n\geq 0} (a_n + b_n)(z-z_0)^n \\ c\cdot f(z) &= \sum_{n\geq 0} c\cdot a_n (z-z_0)^n \end{aligned}\right\} \text{Taylor series expansions}$$

Ex1: Find radius of convergence of

a) $\sum_{n\geq 1} z^{n!}$, b) $\sum_{n\geq 1} \frac{z^n}{n^2}$, c) $\sum_{n\geq 1} \frac{z^n}{n}$

a) If using Hadamard f-la, then $a_n = \begin{cases} 1 & \text{if } n=k! \\ 0 & \text{else} \end{cases} \Rightarrow \sqrt[n]{a_n} = \begin{cases} 1, & \text{if } n=k! \\ 0, & \end{cases}$

$\Rightarrow R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 1$.

If using ratio test: $\frac{z^{(n+1)!}}{z^{n!}} = z^{(n+1)!-n!} \xrightarrow{\text{if } |z|<1} 0 \Rightarrow$ R.O.C ≥ 1

but for $|z|\geq 1$: $|z^{n!}| \geq 1 \Rightarrow$ diverges

$\Rightarrow \boxed{R=1}$

b) Ratio test: $\frac{z^{n+1}}{(n+1)^2} \cdot \frac{n^2}{z^n} = \left(\frac{n}{n+1}\right)^2 z \xrightarrow{n\to\infty} z \Rightarrow \boxed{R=1}$

c) Same reasoning $\Rightarrow \boxed{R=1}$

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Ex 2: In each of the series from Ex 1 describe if there is no, all, or some points on convergence circle where sum converges.

a) If $|z|=1 \Rightarrow |z^{n!}|=1 \ \forall n \Rightarrow$ diverges at all points on unit circle.

b) If $|z|=1 \Rightarrow \left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$ and $\sum_{n\geq 1} \frac{1}{n^2}$ converges $\Rightarrow$ converges at all points on unit circle.

c) If $z=1 \Rightarrow \sum_{n\geq 1} \frac{z^n}{n} = \sum_{n\geq 1} \frac{1}{n}$ diverges
If $z=-1 \Rightarrow \sum_{n\geq 1} \frac{(-1)^n}{n}$ - converges
$\Rightarrow$ converges at some but not all points on unit circle.

Ex 3: Find radius of convergence of $\sum_{n\geq 0} 3^n z^{n^2}$.

Let's illustrate both ratio test & Hadamard formula.

Solution #1 (via Ratio test)

Our series is $1+3z+9z^4+27z^9+\dots$ and consecutive ratios are

$$\frac{3^{n+1} z^{(n+1)^2}}{3^n z^{n^2}} = 3z^{2n+1}.$$

If $|z|<1 \Rightarrow 3z^{2n+1} \xrightarrow{n\to\infty} 0 \Rightarrow$ converges
If $|z|>1 \Rightarrow |3^n z^{n^2}|>1 \ \forall n \Rightarrow$ diverges
$\Rightarrow \boxed{R.O.C = 1}$

Solution #2 (via Hadamard Formula)

It's key to write this series as $\sum a_n z^n = 1+3z+0\cdot z^2+0\cdot z^3+9z^4+0\cdot z^5+\dots$

so that $a_n = \begin{cases} 3^k & \text{if } n=k^2,\ k>0 \\ 0 & \text{else} \end{cases} \Rightarrow \sqrt[n]{a_n} = \begin{cases} \sqrt[k]{3} & \text{if } n=k^2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \limsup = 1$

$\Rightarrow \boxed{R.O.C = 1/1 = 1}$

Note: Common mistake is to apply Hadamard f-la with $a_n = 3^n$, in which case you would get $R.O.C = \frac{1}{3}$

get 0 on exam!

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Ex 4: Find radius of convergence of $\sum_{n\geq 0} a_n z^n$ with

$$a_0 = 1, \quad a_n = \begin{cases} \dfrac{n(n+1)(2n+3)(2n+7)}{n(n+1)(n+3)(n+7)} a_{n-4} & \text{if } 4 \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

► Apply ratio test to $1 + a_4 z^4 + a_8 z^8 + a_{12} z^{12} + \dots$

$$\frac{a_n z^n}{a_{n-4} z^{n-4}} = \frac{n(n+1)(2n+3)(2n+7)}{n(n+1)(n+3)(n+7)} z^4 \xrightarrow[n\to\infty]{} 4z^4$$

So if $\left.\begin{array}{l} |4z^4| < 1 \Rightarrow \text{converges} \\ |4z^4| > 1 \Rightarrow \text{diverges} \end{array}\right\} \Rightarrow R.O.C = 1/\sqrt{2}$

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Ex 5: Find radius of convergence of $\sum_{n\geq 0} (2^n + 3^n) z^n$.

► One can apply ratio test again to conclude $R.O.C = 1/3$.

(!) Another way is to note that $\sum_{n\geq 0} 2^n z^n = \sum (2z)^n$ has $R.O.C = 1/2$ while $\sum_{n\geq 0} 3^n z^n = \sum_{n\geq 0} (3z)^n$ has $R.O.C = 1/3$ $\Rightarrow$ their sum has $R.O.C = \min\{\frac{1}{3}, \frac{1}{2}\} = \frac{1}{3}$ (use Taylor series expansion to interpret its limit as $\frac{1}{1-2z} + \frac{1}{1-3z}$)

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Note: However if $\sum a_n z^n$ & $\sum b_n z^n$ have the same radius of convergence R, then $\sum (a_n + b_n) z^n$ has radius of convergence $\geq R$.

e.g. $\sum_{n\geq 1} z^n$ has $R.O.C = 1$
$\sum_{n\geq 1} (\frac{1}{n!} - 1) z^n$ has $R.O.C = 1$

BUT their sum $\sum_{n\geq 1} \frac{z^n}{n!}$ has $R.O.C = \infty$

In all but the last problem there was not much of complex analysis used. But we can always ask which analytic limit $f(z)$ represents the power series $\sum a_n (z - z_0)^n$ inside convergence disk. But this is also how sometimes R.O.C. is found ↘

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Ex6: Let $\{F_n\}_{n=0}^{\infty}$ be the famous Fibonacci sequence, i.e. $F_0=1$, $F_1=1$, and $F_{n+2}=F_{n+1}+F_n$ for any $n\geq 0$. Find R.O.C. of $\sum_{n=0}^{\infty} F_n z^n$.

▷

$$\left.\begin{aligned} \text{Let}\quad F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \dots \\ \text{Then}\quad zF(z) &= \qquad F_0 z + F_1 z^2 + F_2 z^3 + \dots \\ z^2F(z) &= \qquad\qquad F_0 z^2 + F_1 z^3 + \dots \end{aligned}\right\} \Rightarrow F(z)(1-z-z^2)=1$$

So: $F(z)$ is a series inverse to polynomial $1-z-z^2$

Let's look at the function $\frac{1}{1-z-z^2}$ on $\mathbb{C}$: it's analytic except for two roots of polynomial z^2+z-1, which are $\frac{-1\pm\sqrt{5}}{2}$. Since $\left|\frac{-1+\sqrt{5}}{2}\right| < \left|\frac{-1-\sqrt{5}}{2}\right|$ we see that Taylor disk has radius $\frac{\sqrt{5}-1}{2}$ and coefficients of the Taylor expansion of $\frac{1}{1-z-z^2}$ satisfy same conditions as Fibonacci numbers. Thus: $\boxed{R=\frac{\sqrt{5}-1}{2}}$

Note: Performing partial fraction decomposition to $\frac{1}{1-z-z^2}$ and evoking geometric progression formula $\frac{1}{1-a\cdot z} = 1+az+a^2z^2+a^3z^3+\dots$, one derives explicit formula for Fibonacci numbers (see Ex 10 on p. 268 of textbook)

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$