Lecture #24

Ex1: Find R.O.C. of:

a) $\sum_{n\geq 1} \frac{n!}{10^n} z^n$

b) $\sum_{n\geq 1} \frac{n!}{2^{n^2}} z^n$

c) $\sum_{n\geq 1} \frac{n^n}{n!} z^n$

► We shall use Ratio test for all three series.

a) $c_n = \frac{n!}{10^n} z^n \Rightarrow \frac{c_{n+1}}{c_n} = \frac{n+1}{10} z \xrightarrow[n\to\infty]{} \infty$ for any $z \neq 0 \Rightarrow \boxed{R=0}$

b) $c_n = \frac{n!}{2^{n^2}} z^n \Rightarrow \frac{c_{n+1}}{c_n} = \frac{n+1}{2^{2n+1}} z \xrightarrow[n\to\infty]{} 0$ for any $z \Rightarrow \boxed{R=\infty.}$

c) $c_n = \frac{n^n}{n!} z^n \Rightarrow \frac{c_{n+1}}{c_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} z = \left(1+\frac{1}{n}\right)^n z \xrightarrow[n\to\infty]{} e\cdot z \Rightarrow \boxed{R = 1/e}$

Rmk: If using Hadamard formula you may need to utilize famous

$$\boxed{\text{Stirling Formula } \sqrt[n]{n!} \sim \frac{n}{e}(2\pi n)^{1/n}}$$

Ex2: Find Maclaurin series of

a) $f(z) = \frac{z+z^2}{1-2z}$

b) $f(z) = \frac{1}{2-3z+z^2}$

► a) $\frac{1}{1-z} = 1+z+z^2+z^3+\dots \xrightarrow{\text{Plug } 2z} \frac{1}{1-2z} = 1+2z+4z^2+8z^3+__$

$\Rightarrow f(z) = (z+z^2)(1+2z+4z^2+8z^3+__) \xrightarrow{\text{open brackets}} z + \sum_{n=2}^{\infty} 3\cdot 2^{n-2} z^n$

b) Use partial fraction decomposition first: $f(z) = \frac{1}{1-z} - \frac{1}{2-z}$

Know: $\frac{1}{1-z} = 1+z+z^2+z^3+__$

$\frac{1}{2-z} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}z} = \frac{1}{2}\left(1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+__\right)$

$\Rightarrow f(z) = \sum_{n=0}^{\infty} \left(1 - 1/2^{n+1}\right) z^n$

Lecture #24

* First, let's start with a couple of questions related to already covered §5.2-3.

$\boxed{Q}$: If f, g are analytic at nbhd of z_0 with Taylor series

$$f(z) = \sum_{j\geq 0} a_j (z-z_0)^j, \quad g(z) = \sum_{j\geq 0} b_j (z-z_0)^j,$$

what is the Taylor series of $f(z)\cdot g(z)$?

► Easy to see that $(fg)^{(n)} = \sum_{r=0}^{n} f^{(n-r)} \cdot g^{(r)} \cdot \frac{n!}{r!(n-r)!} \Rightarrow$

$$\Rightarrow \sum_{n\geq 0} \frac{(fg)^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n\geq 0}\sum_{r=0}^{n} \underbrace{\frac{f^{(n-r)}(z_0)}{(n-r)!}}_{=a_{n-r}} (z-z_0)^{n-r} \cdot \underbrace{\frac{g^{(r)}(z_0)}{r!}}_{=b_r} (z-z_0)^r$$

So: coeff. of $(z-z_0)^n$ is $(a_n b_0 + a_{n-1} b_1 + __ + a_0 b_{n-1})$ ■

[Note: The above gives a natural product operation on power series:

$$(a_0 + a_1 z + a_2 z^2 + \dots)\cdot(b_0 + b_1 z + b_2 z^2 + __) = (a_0 b_0) + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \dots$$

Ex 3: Find the first 4 terms of the Maclaurin series of $\tan z$.

► $$\underbrace{\tan z}_{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots} = \frac{\sin z}{\cos z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots}$$

$$\Rightarrow \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + __\right) = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 - \frac{1}{2} a_0 = 0 \Rightarrow a_2 = 0 \\ a_3 - \frac{a_1}{2} = -\frac{1}{6} \Rightarrow a_3 = \frac{1}{3} \\ \vdots \end{cases} \Rightarrow \boxed{\tan z = z + \frac{z^3}{3} + \underbrace{\text{h.o.t.}}_{\text{higher powers of } z}}$$

[Note: Computing explicitly $\tan^{(n)}(0)$ is quite cumbersome!!!

Lecture #24

- As discussed in Examples 1-2 of Section 5.3, one can use Taylor series to solve O.D.E.

Ex 4 (= Problem 13(a) on p. 260): Find an analytic solution of O.D.E

$$f''(z) - z f'(z) - f(z) = 0 \quad \text{with initial condition} \quad f(0) = 1,\ f'(0) = 0.$$

[Hint: Determine recursion on coefficients of Maclaurin series of $f(z)$.

Let $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$

Then: $f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots$

$f''(z) = 2a_2 + 6a_3 z + 12 a_4 z^2 + \dots$

So: $\underbrace{f''(z) - z f'(z) - f(z)}_{=0} = \sum_{j=0}^{\infty} \left((j+1)(j+2)a_{j+2} - j a_j - a_j\right) z^j$

$\Rightarrow (j+1)\left((j+2)a_{j+2} - a_j\right) = 0$ for all j!

$\Rightarrow \boxed{a_{j+2} = \frac{1}{j+2} a_j}$

Initial conditions $\Rightarrow a_0 = 1$, $a_1 = 0$

$$\Rightarrow a_j = \begin{cases} 0, & j - \text{odd} \\ \frac{1}{2 \cdot 4 \cdots 2k}, & j = 2k \end{cases}$$

So: $f(z) = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} z^{2k} = \sum_{k=0}^{\infty} \frac{(z^2/2)^k}{k!} = \boxed{e^{z^2/2}}$.

[Interested Students: Example of Bessel function of the 1st kind of order 0 (see p. 266 of your textbook)

Lecture #24 (§5.6)

For the rest of today we shall discuss zeros of analytic functions.

Let f be an analytic function in nbhd of $z_0 \in \mathbb{C}$.

Def: A point $z_0 \in \mathbb{C}$ is called a zero of order m for $f(z)$ if

$$f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0.$$

Thus, the Taylor series of $f(z)$ has the form:

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots = (z-z_0)^m \cdot \left(a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \dots\right)$$

Note: The series $a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \dots$ converges whenever $a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + a_{m+2}(z-z_0)^{m+2} + \dots$ does, hence, it converges in some nbhd of z_0.

So: $f(z) = (z-z_0)^m \cdot g(z)$ and $g(z_0) = a_m \neq 0$

analytic in some $D_R(z_0)$

Proposition 1: Let f be analytic in nbhd of z_0. Then f has a zero of order m at z_0 iff $\frac{f(z)}{(z-z_0)^m} =: g(z)$ is analytic in nbhd of z_0 and $g(z_0) \neq 0$.

As an immediate consequence (based on above $g(z)$ being continuous):

Lemma 1. If f is analytic in $D_R(z_0)$ and $f(z_0) = 0$, then either $f \equiv 0$ on $D_R(z_0)$ or there is $r > 0$ s.t. f is not vanishing on $D_r(z_0) \setminus \{z_0\}$

This can be generalized to:

Theorem 1: Suppose f is analytic in a domain $\Omega \subseteq \mathbb{C}$. Let Z_f be the zero set $Z_f = \{z \in \Omega \mid f(z) = 0\}$. If Z_f has a limit point $z_0 \in \Omega$ (i.e. $\exists\, z_1, z_2, \dots \in Z_f$, $z_i \neq z_0$, $z_i \xrightarrow[i\to\infty]{} z_0$), then $f \equiv 0$ on Ω.

Lecture #24

Proof of Thm 1

The proof is very similar to the proof of Theorem 3 in Lecture 17.

Assume $z_0 \in \Omega$ is the limit point of Z_f. By Lemma 1, we then know $f \equiv 0$ on some disk $D_R(z_0)$. <u>Want</u>: show that $f \equiv 0$ on Ω.

For any $z_1 \in \Omega \setminus z_0$, find a continuous path γ $z_0 \to z_1$ and parametrize it $z : [0,1] \to \gamma$.

Let $\tau = \sup\{0 \le t \le 1 \mid f(z(s)) = 0 \ \forall\, 0 \le s < t\}$

As f-continuous $\Rightarrow$ above $\sup\{\dots\} = \max\{\dots\}$, i.e. $f(z(\tau)) = 0$.

Also $\tau > 0$ as we know $f|_{D_R(z_0)} \equiv 0$. But applying Lemma 1 to $z(\tau)$-limit point of Z_f, we conclude that $f \equiv 0$ on some disk $D_r(z(\tau))$.

<u>Thus</u>: if $\tau < 1$, then $\exists \varepsilon > 0$ s.t. $f(z(t)) = 0 \ \forall\, 0 \le t < \tau + \varepsilon \Rightarrow$
$\Rightarrow$ contradiction with choice of τ

<u>So</u>: $\tau = 1 \Rightarrow f(z_1) = 0$. ∎

<u>Warning</u>: It's important that limit point was assumed to be in Ω too. E.g. consider $f(z) = \sin\frac{1}{z}$ – analytic on $\mathbb{C} \setminus \{0\}$. Then, the zeros $\{\frac{1}{\pi k} \mid k > 0 \text{ integer}\}$ of f have a limit point 0, <u>but</u> f is not defined at 0! (hence theorem above doesn't apply!!)

<u>Note</u>: Above thm fails for C^∞-smooth f-s on $\mathbb{R}$! E.g. $f(t) = \begin{cases} e^{-1/t^2} \sin\frac{1}{t}, & t \ne 0 \\ 0, & t = 0 \end{cases}$
It has a limit point of its zeros ($z_0 = 0$).

Lecture #24

Corollary 1: Any differentiable function $f(x)$ on $\mathbb{R}$ (or even $[a,b] \subseteq \mathbb{R}$, $a \neq b$) can be extended to an entire function $f(z)$ in at most one way!

▸ Indeed if $F_1(z), F_2(z)$ are two such extensions, then $F_1(z) - F_2(z)$ is entire and vanishes on $\mathbb{R}$ (or $[a,b]$) $\Rightarrow$ its zeros have limit point $\overset{\text{Thm 1}}{\Longrightarrow}$ $F_1(z) = F_2(z)$ $\forall z \in \mathbb{C}$. ∎

- Examples: $e^x, \sin x, \cos x$ can be extended at most uniquely (and we found such extensions long time ago!).

Corollary 2: Any functional equalities $/\mathbb{R}$ would still hold for analytic extensions

This is less precise, so let's just illustrate with examples:

- Example: $\cos^2 z + \sin^2 z = 1$ $\forall z \in \mathbb{C}$

▸ Indeed, $\cos^2 z + \sin^2 z - 1$ is entire, vanishing on $\mathbb{R}$ $\Rightarrow$ vanishes on $\mathbb{C}$ ∎

- Example: $\cos(z+w) = \cos z \cdot \cos w - \sin z \cdot \sin w$ $\forall z, w \in \mathbb{C}$

▸ Fix any $z \in \mathbb{R}$. Then $g(w) := \cos(z+w) - \cos z \cdot \cos w + \sin z \cdot \sin w$ is entire in $w \in \mathbb{C}$, vanishing for $w \in \mathbb{R}$ $\Rightarrow$ $g(w) = 0$ $\forall w$.

Now for any $w \in \mathbb{C}$, fix it, and consider above as f-n in $z \in \mathbb{C}$. It vanishes on $\mathbb{R}$ by above $\Rightarrow$ zero everywhere! ∎