Lecture #25

Last time we discussed zeros of analytic functions. We proved first:

Lemma 1*: If $f(z)$ - analytic in $D_R(z_0)$ and $f(z_0)=0$, then either $f(z)\equiv 0$ on $D_R(z_0)$ OR there is some $0<r<R$ such that $f(z)\neq 0$ for any $0<|z-z_0|<r$.

and then we generalized this to the case of general domains in $\mathbb{C}$:

Theorem 1*: If $f(z)$ is analytic in a domain $\Omega\subseteq\mathbb{C}$ and the set of zeros $Z_f=\{z\in\Omega : f(z)=0\}$ has a limit point in Ω, then $f(z)\equiv 0$ on Ω.

Ex 1: Does the analogue of Theorem 1* hold for harmonic functions?

► No: the function $u(x,y)=x$ is harmonic but its zeros are $\{(0,y) \mid y\in\mathbb{R}\}$ ■

Ex 2: Is there a harmonic function $u(x,y)$ on $\mathbb{R}^2$ whose zeros set has no limit points and is nonempty?

► No: Assume such u existed and let $u(x_0,y_0)=0$ while $u(x,y)\neq 0$ for all $(x,y)\in D_r(x_0,y_0)\setminus(x_0,y_0)$. By mean-value theorem for harmonic functions: $u(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi} u(\underset{x_0+iy_0}{z_0}+\frac{r}{2}e^{i\theta})d\theta$. But assuming $u(z_0+\frac{r}{2}e^{i\theta})$ is never zero, the integral is either positive or negative $\Rightarrow$ can't be 0. Contradiction! ■

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Claim 1: If a function u is harmonic in a domain Ω in $\mathbb{C}=\mathbb{R}^2$ and vanishes on some open disk $D_\varepsilon(z_0)\subseteq\Omega$, then $u(z)\equiv 0$ on Ω

Note: While similar to Theorem 1* for analytic functions, the same proof completely fails due to above Ex 1-2

▶ The idea is standard by now, see the proof of Theorem 1 in Lecture 18: consider the function $g(z)=u_x-iu_y$ where $z=x+iy$. We note that $g(z)$ satisfies Cauchy-Riemann conditions:

$$\begin{cases} (u_x)_x = (-u_y)_y & \text{because } u(x,y)\text{ - harmonic} \\ (u_x)_y = -(-u_y)_x & \text{is the equality of mixed partials} \end{cases}$$

So: $g(z)$ - analytic on Ω and $g(z)\equiv 0$ on $D_\varepsilon(z_0) \overset{\text{Thm 1*}}{\Longrightarrow} g(z)\equiv 0$ on Ω

$\Rightarrow u_x\equiv 0,\ u_y\equiv 0$ on $\Omega \Rightarrow u\equiv \text{const} \Rightarrow u\equiv 0$ on Ω ■

Before jumping to our next topic, we also note the following:

Claim 2: If $f(z)$ - analytic near z_0, then radius of convergence of its Taylor series equals radius of convergence of Taylor series of f' at z_0

▶ If $f(z)$ is analytic in $D_R(z_0)$, then so is $f'(z) \Rightarrow R_{f'}\geq R_f$.

Assume that $R_{f'}>R_f$, i.e. Taylor series for $f'(z)$ converges to some analytic function $g(z)$ on $D_{R_{f'}}(z_0)$. Then there is an analytic antiderivative $G(z)$ of $g(z)$ on $D_{R_{f'}}(z_0)$, s.t. $G'(z)=f'(z)$ on $D_{R_f}(z_0) \Rightarrow G(z)\equiv f(z)+\underbrace{C}_{\text{constant}}$ on the disk $D_{R_f}(z_0)$

But: the Taylor series of a function $h(z)$ at z_0 only depends on the values of $h(z)$ in any small disk $D_\varepsilon(z_0)$

$\Rightarrow$ Taylor series of $G(z)$ & $f(z)$ differ only in the fixed term $\Rightarrow$ have the same radius of convergence $\Rightarrow$ Contradiction! So: $R_{f'}=R_f$ ■

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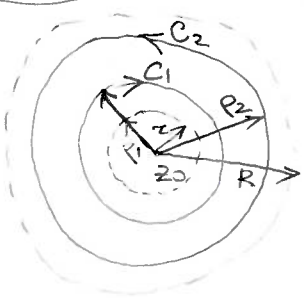
For the rest of today we shall be looking at washer = annulus

$$\Omega = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$

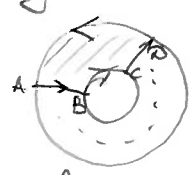
for some $z_0 \in \mathbb{C}$ and real $0 \le r < R$.

Thm 1: For any $r < \rho_1 < \rho_2 < R$, let C_2 = positively oriented circle of radius ρ_2 around z_0 and C_1 = negatively oriented circle of radius ρ_1 around z_0. For any $f(z)$ analytic on above washer/annulus, we have:

$$\oint_{C_2} f(z)\,dz + \oint_{C_1} f(z)\,dz = 0$$



▸ This is an immediate consequence of Cauchy theorem, as discussed a few weeks ago. Easy way to see it is to split into 2 domains and apply Cauchy thm to each, and then add those



$$\int_{\overrightarrow{AB}} + \int_{\overrightarrow{BA}} = 0 = \int_{\overrightarrow{AB}} + \int_{\overrightarrow{BC}}$$

∎

In a completely similar fashion, the Cauchy integral formula implies the following result:

Thm 2: For any $\rho_1 < |a - z_0| < \rho_2$, we have

$$f(a) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-a}\,dz + \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a}\,dz$$

We shall now use this result to represent $f(z)$, analytic on the above annulus $r < |z - z_0| < R$, by a power series <u>BUT</u> involving now both positive and <u>negative</u> powers of $z - z_0$.

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Theorem 3 (main result today!): Let $f(z)$ be analytic in $r<|z-z_0|<R$. Then:

$$f(z)=\sum_{j=0}^{\infty} a_j(z-z_0)^j+\sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} \quad \text{s.t.:}$$

a) both sums converge in above annulus

b) both sums converge uniformly in any subannulus $\tilde{\rho}_1\le|z-z_0|\le\tilde{\rho}_2$ for any $r<\tilde{\rho}_1<\tilde{\rho}_2<R$

Moreover, $a_j=\frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{j+1}}\,dw$ for all $j\in\mathbb{Z}$ and any positively oriented simple closed curve C in annulus with z_0 inside it

[Note: This should be compared with Theorem 1 of Lecture 22, where we represented a function $f(z)$ analytic in $D_R(z_0)$ by its Taylor series.

▶ • Given any z in above $\Omega=\{z\in\mathbb{C}: r<|z-z_0|<R\}$, we can take $\tilde{\rho}_1=\frac{r+|z-z_0|}{2}$, $\tilde{\rho}_2=\frac{R+|z-z_0|}{2}$, so that b) $\Rightarrow$ a). Let's now prove b).

• Given $r<\tilde{\rho}_1<\tilde{\rho}_2<R$, we shall pick ρ_1,ρ_2 s.t. $r<\rho_1<\tilde{\rho}_1<\tilde{\rho}_2<\rho_2<R$, e.g. similarly to Lecture 22, we shall take $\rho_1=\frac{r+\tilde{\rho}_1}{2}$, $\rho_2=\frac{R+\tilde{\rho}_2}{2}$, and let C_1, C_2 be the circles as in Thms 1&2. We shall now apply Theorem 2 from above:

$$\boxed{f(z)=\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw+\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw}$$

Following word-to-word proof of Thm 1 of Lecture 22, based on

$$\frac{1}{w-z}=\frac{1}{(w-z_0)-(z-z_0)}=\frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\frac{(z-z_0)^2}{(w-z_0)^3}+\dots,$$

we get

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw=\sum_{j=0}^{n}a_j(z-z_0)^j+\mathcal{E}_n(z) \quad \text{with}$$

$\mathcal{E}_n(z)\xrightarrow[n\to\infty]{}0$ uniformly in $\tilde{\rho}_1\le|z-z_0|\le\tilde{\rho}_2$

$$a_j=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-z_0)^{j+1}}\,dw$$

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(Continuation)

To evaluate $\int_{C_1} \frac{f(w)}{w-z}\,dw$ we rather rewrite

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = -\frac{1}{z-z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}} = -\frac{1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^j \quad \text{as } \left|\frac{w-z_0}{z-z_0}\right|<1$$

for $w\in C_1$

Fix any $n\geq 0$ and look at partial sum above, so that

$$\frac{1}{w-z} = -\frac{1}{z-z_0} - \frac{w-z_0}{(z-z_0)^2} - \dots - \frac{(w-z_0)^n}{(z-z_0)^{n+1}} + \frac{1}{w-z}\cdot\left(\frac{w-z_0}{z-z_0}\right)^{n+1}$$

So: $\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = a_{-1}\cdot(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + ___ + a_{-n-1}(z-z_0)^{-n-1} + \tilde{\mathcal{E}}_n(z)$,

where $a_{-j} = -\frac{1}{2\pi i}\oint_{C_1} f(w)(w-z_0)^{j-1}\,dw$ for $j\geq 1$

$$= \frac{1}{2\pi i}\oint_{-C_1}\frac{f(w)}{(w-z_0)^{-j+1}}\,dw$$

same circle but now positively oriented

and

$$\tilde{\mathcal{E}}_n(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\left(\frac{w-z_0}{z-z_0}\right)^{n+1}dw.$$

But we can estimate this integral as before:

$$\left.\begin{array}{l} |f(w)|\leq M \ \forall w\in C_1 \ (M\text{- some big upper bound}) \\ |w-z|\geq \tilde{\rho}_1-\rho_1 \Rightarrow \frac{1}{|w-z|}\leq\frac{1}{\tilde{\rho}_1-\rho_1} \\ \left|\frac{w-z_0}{z-z_0}\right|\leq\frac{\rho_1}{\tilde{\rho}_1}<1 \end{array}\right\} \Rightarrow |\tilde{\mathcal{E}}_n(z)|\leq\frac{1}{2\pi}\cdot 2\pi\rho_1\cdot M\cdot\frac{1}{\tilde{\rho}_1-\rho_1}\left(\rho_1/\tilde{\rho}_1\right)^n$$

$$\Downarrow$$

$\tilde{\mathcal{E}}_n(z)\xrightarrow[n\to\infty]{}0$ uniformly on the annulus $\tilde{\rho}_1\leq|z-z_0|\leq\tilde{\rho}_2$.

The rest of the theorem follows now ∎

Note: Unlike in Lecture 22, we are <u>no longer</u> saying that a_j equals $\frac{1}{j!}f^{(j)}(z_0)$ for $j\geq 0$ (as the latter may be not defined even)

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Def: The above sum $\sum_{j=-\infty}^{+\infty} a_j (z-z_0)^j$ is called Laurent series of $f(z)$

We also have the following analogue of Theorem 3 from Lecture 22:

Theorem 4: If $\sum_{j=0}^{+\infty} c_j (z-z_0)^j$ converges inside $C_R(z_0)$

$\sum_{j=1}^{\infty} c_{-j} (z-z_0)^{-j}$ converges outside $C_r(z_0)$

and $r<R$, then there is an analytic function $f(z)$ in the washer $r<|z-z_0|<R$ whose Laurent series in it is exactly $\sum_{j=0}^{+\infty} c_j (z-z_0)^j + \sum_{j=1}^{\infty} c_{-j} (z-z_0)^{-j}$.

We shall skip the proof for now (as it's also omitted in the book) see Theorem 15 on page 273 of textbook

Note. If $r=0$, then the annulus above is just $\boxed{D_R(z_0)\setminus\{z_0\} = \text{punctured open disk}}$

Def: z_0 – isolated singularity of $f(z)$ if $f(z)$ is analytic on $D_\varepsilon(z_0)\setminus\{z_0\}$ (for some $\varepsilon>0$) but is not analytic at z_0 itself

Looking at the Laurent series of $f(z)$ in $0<|z-z_0|<R$, we shall distinguish 3 cases:

Def: a) If $a_{<0}=0$, then z_0 is called removable singularity.

b) If $a_{-m}\neq 0$ & $a_{<-m}=0$ for some $m>0$, then z_0 is called a pole of order m

c) If $a_{-j}\neq 0$ for ∞ many $j>0$, then z_0 – essential singularity

Examples: $\frac{z^2-1}{z-1}$ has $z_0=1$ as removable singularity

$\frac{\sin z}{z}$ has $z_0=0$ as removable singularity

$\frac{\sin z}{z^3}$ has $z_0=0$ as a pole of order 2

$e^{1/z}$ has $z_0=0$ as essential singularity